Chapter 4 Applications of Derivatives

An automobile's gas mileage is a function of many variables, including road surface, tire type, velocity, fuel octane rating, road angle, and the speed and direction of the wind. If we look only at velocity's effect on gas mileage, the mileage of a certain car can be approximated by:

$$m(v) = 0.00015v^3 - 0.032v^2 + 1.8v + 1.7$$

(where v is velocity)

At what speed should you drive this car to obtain the best gas mileage? The ideas in Section 4.1 will help you find the answer.

Chapter 4 Overview

In the past, when virtually all graphing was done by hand—often laboriously—derivatives were the key tool used to sketch the graph of a function. Now we can graph a function quickly, and usually correctly, using a grapher. However, confirmation of much of what we see and conclude true from a grapher view must still come from calculus.

This chapter shows how to draw conclusions from derivatives about the extreme values of a function and about the general shape of a function's graph. We will also see how a tangent line captures the shape of a curve near the point of tangency, how to deduce rates of change we cannot measure from rates of change we already know, and how to find a function when we know only its first derivative and its value at a single point. The key to recovering functions from derivatives is the Mean Value Theorem, a theorem whose corollaries provide the gateway to *integral calculus,* which we begin in Chapter 5.

4.1 Extreme Values of Functions

What you'll learn about

- Absolute (Global) Extreme Values
- Local (Relative) Extreme Values
- Finding Extreme Values

. . . and why

Finding maximum and minimum values of functions, called optimization, is an important issue in real-world problems.

Absolute (Global) Extreme Values

One of the most useful things we can learn from a function's derivative is whether the function assumes any maximum or minimum values on a given interval and where these values are located if it does. Once we know how to find a function's extreme values, we will be able to answer such questions as "What is the most effective size for a dose of medicine?" and "What is the least expensive way to pipe oil from an offshore well to a refinery down the coast?" We will see how to answer questions like these in Section 4.4.

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) absolute maximum value on D if and only if $f(x) \le f(c)$ for all x in D.

(b) absolute minimum value on D if and only if $f(x) \ge f(c)$ for all x in D.

Absolute (or **global**) maximum and minimum values are also called **absolute extrema** (plural of the Latin *extremum*). We often omit the term "absolute" or "global" and just say maximum and minimum.

Example 1 shows that extreme values can occur at interior points or endpoints of intervals.

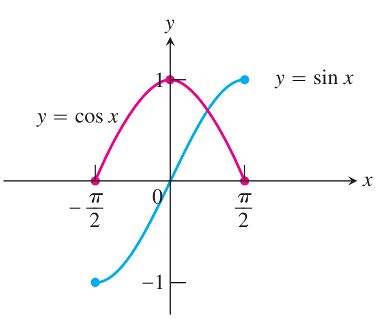


Figure 4.1 (Example 1)

EXAMPLE 1 Exploring Extreme Values

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1). ***Now try Exercise 1.***

Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 2 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d)	$y = x^2$	$(0, 2)$	No absolute extrema.

Now try Exercise 3.

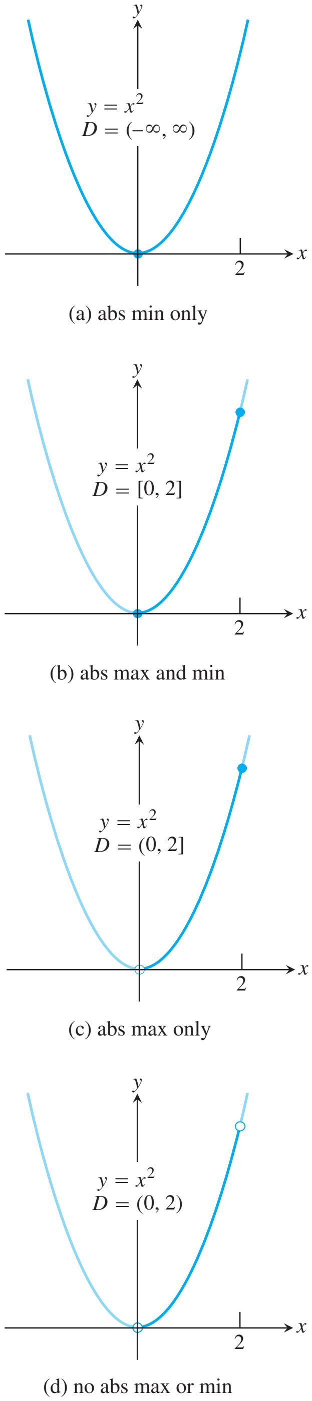


Figure 4.2 (Example 2)

Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

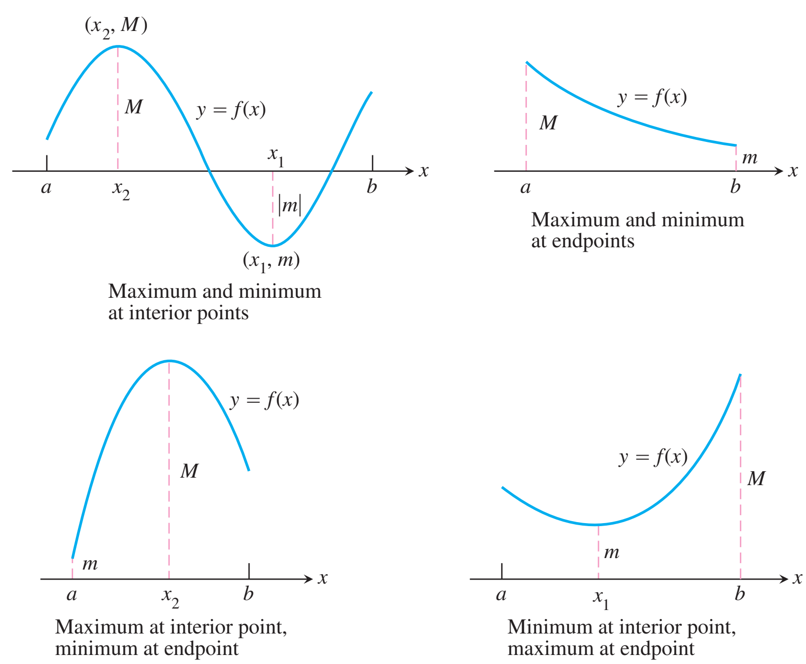


Figure 4.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

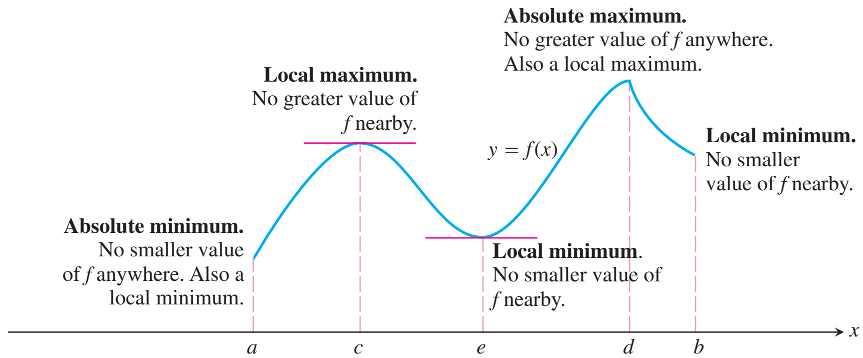


Figure 4.4 Classifying extreme values.

Local (Relative) Extreme Values

Figure 4.4 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITION **Local Extreme Values**

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Local extrema are also called **relative extrema.**

An **absolute extremum** is also a local extremum, because being an extreme value overall makes it an extreme value in its immediate neighborhood. Hence, *a list of local extrema will automatically include absolute extrema if there are any.*

Finding Extreme Values

The interior domain points where the function in Figure 4.4 has local extreme values are points where either f' is zero or f' does not exist. This is generally the case, as we see from the following theorem.

THEOREM 2 **Local Extreme Values**

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

Because of Theorem 2, we usually need to look at only a few points to find a function's extrema. These consist of the interior domain points where $f' = 0$ or f' does not exist (the domain points covered by the theorem) and the domain endpoints (the domain points not covered by the theorem). At all other domain points, $f' > 0$ or $f' < 0$.

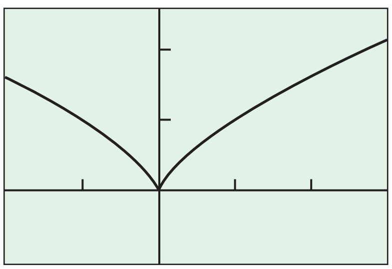
The following definition helps us summarize these findings.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Thus, in summary, extreme values occur only at critical points and endpoints.

$y = x^{2/3}$



[–2, 3] by [–1, 2.5]

Figure 4.5 (Example 3)

EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

SOLUTION

Solve Graphically Figure 4.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Confirm Analytically We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

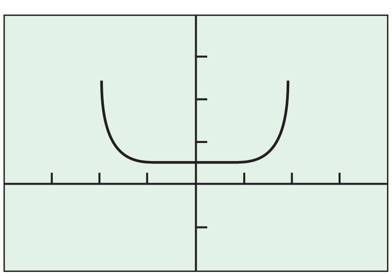
Critical point value: $f(0) = 0$;

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$;

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$. ***Now try Exercise 11.***

In Example 4, we investigate the reciprocal of the function whose graph was drawn in Example 3 of Section 1.2 to illustrate "grapher failure."



[–4, 4] by [–2, 4]

Figure 4.6 The graph of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

(Example 4)

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

SOLUTION

Solve Graphically Figure 4.6 suggests that f has an absolute minimum of about 0.5 at $x = 0$. There also appear to be local maxima at $x = -2$ and $x = 2$. However, f is not defined at these points and there do not appear to be maxima anywhere else.

continued

Confirm Analytically The function f is defined only for $4 - x^2 > 0$, so its domain is the open interval $(-2, 2)$. The domain has no endpoints, so all the extreme values must occur at critical points. We rewrite the formula for f to find f':

$$f(x) = \frac{1}{\sqrt{4 - x^2}} = (4 - x^2)^{-1/2}.$$

Thus,

$$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x) = \frac{x}{(4 - x^2)^{3/2}}.$$

The only critical point in the domain $(-2, 2)$ is $x = 0$. The value

$$f(0) = \frac{1}{\sqrt{4 - 0^2}} = \frac{1}{2}$$

is therefore the sole candidate for an extreme value.

To determine whether $1/2$ is an extreme value of f, we examine the formula

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

As x moves away from 0 on either side, the denominator gets smaller, the values of f increase, and the graph rises. We have a minimum value at $x = 0$, and the minimum is absolute.

The function has no maxima, either local or absolute. This does not violate Theorem 1 (The Extreme Value Theorem) because here f is defined on an *open* interval. To invoke Theorem 1's guarantee of extreme points, the interval must be closed.

Now try Exercise 25.

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figure 4.7 illustrates this for interior points. Exercise 55 describes a function that fails to assume an extreme value at an endpoint of its domain.

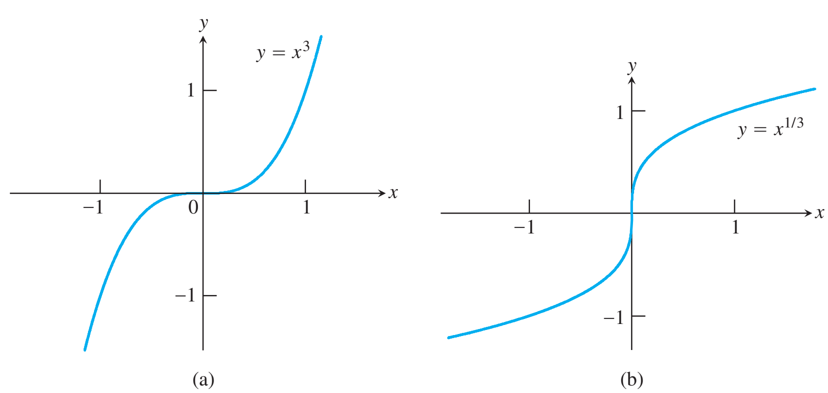


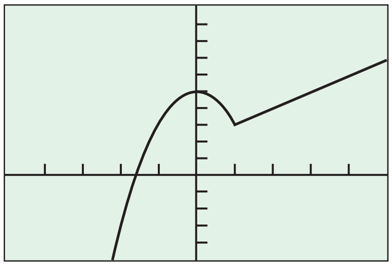
Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

continued



[–5, 5] by [–5, 10]

Figure 4.8 The function in Example 5.

SOLUTION

Solve Graphically The graph in Figure 4.8 suggests that $f'(0) = 0$ and that $f'(1)$ does not exist. There appears to be a local maximum value of 5 at $x = 0$ and a local minimum value of 3 at $x = 1$.

Confirm Analytically For $x \neq 1$, the derivative is

$$f'(x) = \begin{cases} \dfrac{d}{dx}(5 - 2x^2) = -4x, & x < 1 \\ \dfrac{d}{dx}(x + 2) = 1, & x > 1. \end{cases}$$

The only point where $f' = 0$ is $x = 0$. What happens at $x = 1$?

At $x = 1$, the right- and left-hand derivatives are respectively

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h) + 2 - 3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{5 - 2(1 + h)^2 - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h(2 + h)}{h} = -4.$$

Since these one-sided derivatives differ, f has no derivative at $x = 1$, and 1 is a second critical point of f.

The domain $(-\infty, \infty)$ has no endpoints, so the only values of f that might be local extrema are those at the critical points:

$$f(0) = 5 \quad \text{and} \quad f(1) = 3.$$

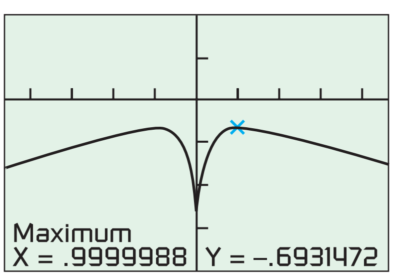
From the formula for f, we see that the values of f immediately to either side of $x = 0$ are less than 5, so 5 is a local maximum. Similarly, the values of f immediately to either side of $x = 1$ are greater than 3, so 3 is a local minimum. ***Now try Exercise 41.***

Most graphing calculators have built-in methods to find the coordinates of points where extreme values occur. We must, of course, be sure that we use correct graphs to find these values. The calculus that you learn in this chapter should make you feel more confident about working with graphs.

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln \left| \dfrac{x}{1 + x^2} \right|$.

SOLUTION



[–4.5, 4.5] by [–4, 2]

Figure 4.9 The function in Example 6.

Solve Graphically The domain of f is the set of all nonzero real numbers. Figure 4.9 suggests that f is an even function with a maximum value at two points. The coordinates found in this window suggest an extreme value of about -0.69 at approximately $x = 1$. Because f is even, there is another extreme of the same value at approximately $x = -1$. The figure also suggests a minimum value at $x = 0$, but f is not defined there.

Confirm Analytically The derivative

$$f'(x) = \frac{1 - x^2}{x(1 + x^2)}$$

is defined at every point of the function's domain. The critical points where $f'(x) = 0$ are $x = 1$ and $x = -1$. The corresponding values of f are both $\ln(1/2) = -\ln 2 \approx -0.69$.

Now try Exercise 37.

EXPLORATION 1 **Finding Extreme Values**

Let $f(x) = \left|\dfrac{x}{x^2 + 1}\right|, -2 \le x \le 2.$

1. Determine graphically the extreme values of f and where they occur. Find f' at these values of x.
2. Graph f and f' (or NDER $(f(x), x, x)$) in the same viewing window. Comment on the relationship between the graphs.
3. Find a formula for $f'(x)$.

Quick Review 4.1 *(For help, go to Sections 1.2, 2.1, 3.5, and 3.6.)*

In Exercises 1–4, find the first derivative of the function.

1. $f(x) = \sqrt{4 - x}$ $\quad \dfrac{-1}{2\sqrt{4-x}}$
2. $f(x) = \dfrac{2}{\sqrt{9 - x^2}}$ $\quad \dfrac{2x}{(9-x^2)^{3/2}}$
3. $g(x) = \cos(\ln x)$ $\quad -\dfrac{\sin(\ln x)}{x}$
4. $h(x) = e^{2x}$ $\quad 2e^{2x}$

In Exercises 5–8, match the table with a graph of $f(x)$.

5. (c)

x	$f'(x)$
a	0
b	0
c	5

6. (b)

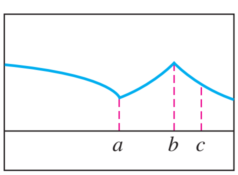
x	$f'(x)$
a	0
b	0
c	-5

7. (d)

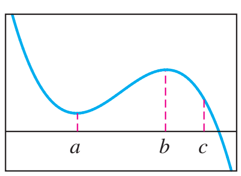
x	$f'(x)$
a	does not exist
b	0
c	-2

8. (a)

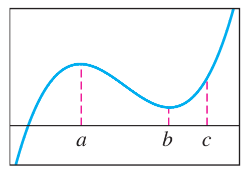
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



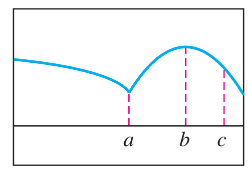
(a)



(b)



(c)



(d)

In Exercises 9 and 10, find the limit for

$$f(x) = \frac{2}{\sqrt{9 - x^2}}.$$

9. $\lim_{x \to 3^-} f(x)$ $\quad \infty$

10. $\lim_{x \to -3^+} f(x)$ $\quad \infty$

In Exercises 11 and 12, let

$$f(x) = \begin{cases} x^3 - 2x, & x \le 2 \\ x + 2, & x > 2. \end{cases}$$

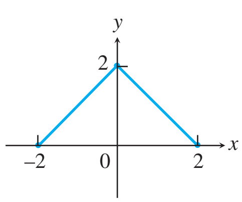
11. Find **(a)** $f'(1)$, 1 **(b)** $f'(3)$, 1 **(c)** $f'(2)$. Undefined

12. **(a)** Find the domain of f'. $\quad x \ne 2$
    (b) Write a formula for $f'(x)$. $\quad f'(x) = \begin{cases} 3x^2 - 2, & x < 2 \\ 1, & x > 2 \end{cases}$

Section 4.1 Exercises

In Exercises 1–4, find the extreme values and where they occur.

1.



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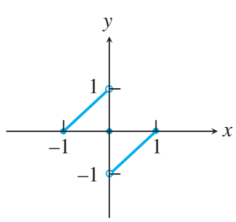

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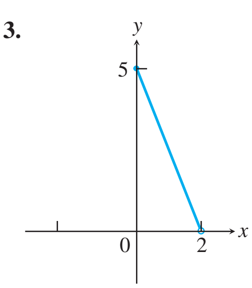

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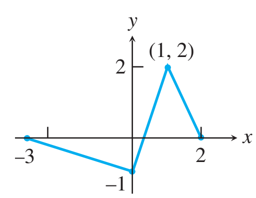
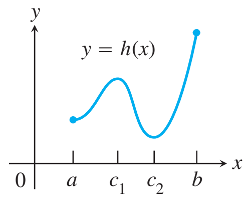


1. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$
2. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$

3. Maximum at $(0, 5)$
4. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$

In Exercises 5–10, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem. See page 195.

5.



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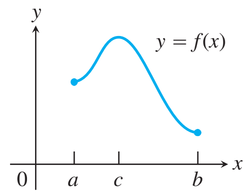

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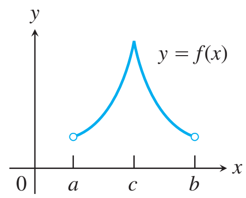

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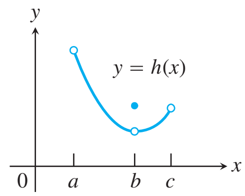
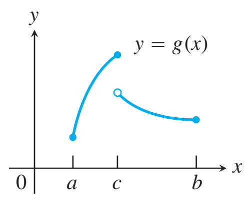
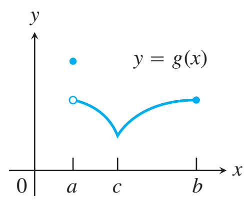


9.



10.



In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur. See page 195.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

12. $g(x) = e^{-x}, \quad -1 \le x \le 1$

13. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \le x \le \dfrac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$ Min value 1 at $x = 2$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

22. $y = x^3 - 3x^2 + 3x - 2$ None

23. $y = \sqrt{x^2 - 1}$ Min value 0 at $x = -1, 1$

24. $y = \dfrac{1}{x^2 - 1}$ Local max at $(0, -1)$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$ Min value 1 at $x = 0$

26. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$ Local min at $(0, 1)$

27. $y = \sqrt{3 + 2x - x^2}$ Max value 2 at $x = 1$; min value 0 at $x = -1, 3$

28. $y = \dfrac{3}{2}x^4 + 4x^3 - 9x^2 + 10$

29. $y = \dfrac{x}{x^2 + 1}$

30. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

21. Local max at $(-2, 17)$; local min at $\left(\dfrac{4}{3}, -\dfrac{41}{27}\right)$

Group Activity In Exercises 31–34, find the extreme values of the function on the interval and where they occur.

31. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

32. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

33. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

34. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

In Exercises 35–42, identify the critical point and determine the local extreme values.

35. $y = x^{2/3}(x + 2)$

36. $y = x^{2/3}(x^2 - 4)$

37. $y = x\sqrt{4 - x^2}$

38. $y = x^2\sqrt{3 - x}$

39. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

40. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

41. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

42. $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

43. Writing to Learn The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

(a) Find the extreme values of V. Max value is 144 at $x = 2$.

(b) Interpret any values found in (a) in terms of volume of the box. The largest volume of the box is 144 cubic units and it occurs when $x = 2$.

44. Writing to Learn The function

$$P(x) = 2x + \frac{200}{x}, \quad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions x by $100/x$.

(a) Find any extreme values of P. Min value is 40 at $x = 10$.

(b) Give an interpretation in terms of perimeter of the rectangle for any values found in (a). The smallest perimeter is 40 units and it occurs when $x = 10$, which makes it a 10 by 10 square.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

45. True or False If $f(c)$ is a local maximum of a continuous function f on an open interval (a, b), then $f'(c) = 0$. Justify your answer.

46. True or False If m is a local minimum and M is a local maximum of a continuous function f on (a, b), then $m < M$. Justify your answer.

47. Multiple Choice Which of the following values is the absolute maximum of the function $f(x) = 4x - x^2 + 6$ on the interval $[0, 4]$? E

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 10

45. False. For example, the maximum could occur at a corner, where $f'(c)$ would not exist.

48. Multiple Choice If f is a continuous, decreasing function on $[0, 10]$ with a critical point at $(4, 2)$, which of the following statements *must be false*? E

(A) $f(10)$ is an absolute minimum of f on $[0, 10]$.

(B) $f(4)$ is neither a relative maximum nor a relative minimum.

(C) $f'(4)$ does not exist.

(D) $f'(4) = 0$

(E) $f'(4) < 0$

49. Multiple Choice Which of the following functions has exactly two local extrema on its domain? B

(A) $f(x) = |x - 2|$

(B) $f(x) = x^3 - 6x + 5$

(C) $f(x) = x^3 + 6x - 5$

(D) $f(x) = \tan x$

(E) $f(x) = x + \ln x$

50. Multiple Choice If an even function f with domain all real numbers has a local maximum at $x = a$, then $f(-a)$ B

(A) is a local minimum.

(B) is a local maximum.

(C) is both a local minimum and a local maximum.

(D) could be either a local minimum or a local maximum.

(E) is neither a local minimum nor a local maximum.

Explorations

In Exercises 51 and 52, give reasons for your answers.

51. Writing to Learn Let $f(x) = (x - 2)^{2/3}$.

(a) Does $f'(2)$ exist? No

(b) Show that the only local extreme value of f occurs at $x = 2$.

(c) Does the result in (b) contradict the Extreme Value Theorem?

(d) Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

52. Writing to Learn Let $f(x) = |x^3 - 9x|$.

(a) Does $f'(0)$ exist? No

(b) Does $f'(3)$ exist? No

(c) Does $f'(-3)$ exist? No

(d) Determine all extrema of f.
Minimum value is 0 at $x = -3$, $x = 0$, and $x = 3$; local maxima at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$

Extending the Ideas

53. *Cubic Functions* Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

(b) How many local extreme values can f have? Two or none

54. *Proving Theorem 2* Assume that the function f has a local maximum value at the interior point c of its domain and that $f'(c)$ exists.

(a) Show that there is an open interval containing c such that $f(x) - f(c) \le 0$ for all x in the open interval.

(b) Writing to Learn Now explain why we may say

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

(c) Writing to Learn Now explain why we may say

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

(d) Writing to Learn Explain how parts (b) and (c) allow us to conclude $f'(c) = 0$.

(e) Writing to Learn Give a similar argument if f has a local minimum value at an interior point.

55. *Functions with No Extreme Values at Endpoints*

(a) Graph the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f.

(b) Group Activity Construct a function of your own that fails to have an extreme value at a domain endpoint.

Answers:

5. Maximum at $x = b$, minimum at $x = c_2$;
Extreme Value Theorem applies, so both the max and min exist.

6. Maximum at $x = c$, minimum at $x = b$;
Extreme Value Theorem applies, so both the max and min exist.

7. Maximum at $x = c$, no minimum;
Extreme Value Theorem doesn't apply, since the function isn't defined on a closed interval.

8. No maximum, no minimum;
Extreme Value Theorem doesn't apply, since the function isn't continuous or defined on a closed interval.

9. Maximum at $x = c$, minimum at $x = a$;
Extreme Value Theorem doesn't apply, since the function isn't continuous.

10. Maximum at $x = a$, minimum at $x = c$;
Extreme Value Theorem doesn't apply, since the function isn't continuous.

11. Maximum value is $\frac{1}{4} + \ln 4$ at $x = 4$; minimum value is 1 at $x = 1$; local maximum at $\left(\frac{1}{2}, 2 - \ln 2\right)$

12. Maximum value is e at $x = -1$; minimum value is $\frac{1}{e}$ at $x = 1$.

13. Maximum value is $\ln 4$ at $x = 3$; minimum value is 0 at $x = 0$.

14. Maximum value is 1 at $x = 0$

15. Maximum value is 1 at $x = \frac{\pi}{4}$; minimum value is -1 at $x = \frac{5\pi}{4}$; local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$; local maximum at $\left(\frac{7\pi}{4}, 0\right)$

16. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$

17. Maximum value is $3^{2/5}$ at $x = -3$; minimum value is 0 at $x = 0$

18. Maximum value is $3^{3/5}$ at $x = 3$

51. (b) The derivative is defined and nonzero for $x \neq 2$. Also, $f(2) = 0$, and $f(x) > 0$ for all $x \neq 2$.
(c) No, because $(-\infty, \infty)$ is not a closed interval.
(d) The answers are the same as (a) and (b) with 2 replaced by a.

4.2 Mean Value Theorem

What you'll learn about

- Mean Value Theorem
- Physical Interpretation
- Increasing and Decreasing Functions
- Other Consequences

. . . and why

The Mean Value Theorem is an important theoretical tool to connect the average and instantaneous rates of change.

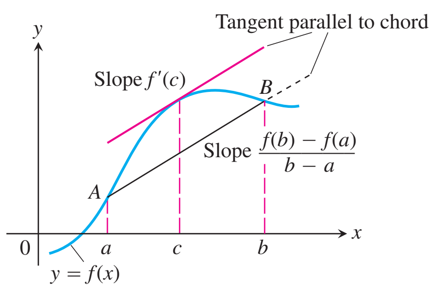


Figure 4.10 Figure for the Mean Value Theorem.

Mean Value Theorem

The Mean Value Theorem connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within the interval. Its powerful corollaries lie at the heart of some of the most important applications of the calculus.

The theorem says that somewhere between points A and B on a differentiable curve, there is at least one tangent line parallel to chord AB (Figure 4.10).

> **THEOREM 3 Mean Value Theorem for Derivatives**
>
> If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The hypotheses of Theorem 3 cannot be relaxed. If they fail at even one point, the graph may fail to have a tangent parallel to the chord. For instance, the function $f(x) = |x|$ is continuous on $[-1, 1]$ and differentiable at every point of the interior $(-1, 1)$ except $x = 0$. The graph has no tangent parallel to chord AB (Figure 4.11a). The function $g(x) = \text{int}\ (x)$ is differentiable at every point of $(1, 2)$ and continuous at every point of $[1, 2]$ except $x = 2$. Again, the graph has no tangent parallel to chord AB (Figure 4.11b).

The Mean Value Theorem is an *existence theorem.* It tells us the number c exists without telling how to find it. We can sometimes satisfy our curiosity about the value of c but the real importance of the theorem lies in the surprising conclusions we can draw from it.

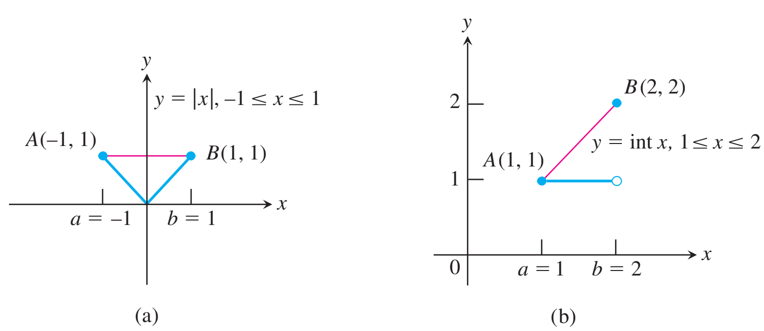
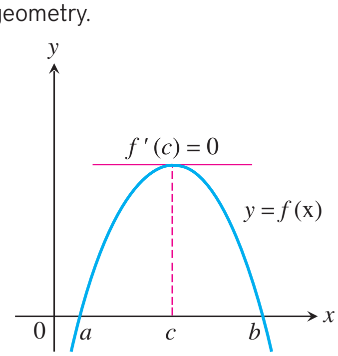


Figure 4.11 No tangent parallel to chord AB.

Rolle's Theorem

The first version of the Mean Value Theorem was proved by French mathematician Michel Rolle (1652–1719). His version had $f(a) = f(b) = 0$ and was proved only for polynomials, using algebra and geometry.



Rolle distrusted calculus and spent most of his life denouncing it. It is ironic that he is known today only for an unintended contribution to a field he tried to suppress.

EXAMPLE 1 Exploring the Mean Value Theorem

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find a solution c to the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

continued

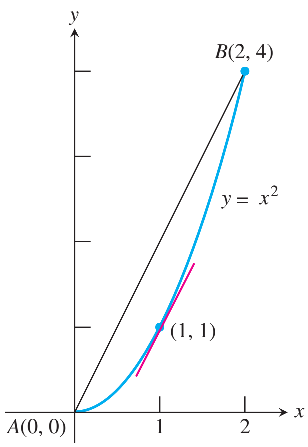


Figure 4.12 (Example 1)

SOLUTION

The function $f(x) = x^2$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem guarantees a point c in the interval $(0, 2)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c = \frac{f(2) - f(0)}{2 - 0} = 2 \qquad f'(x) = 2x$$

$$c = 1.$$

Interpret The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to the chord joining $A(0, 0)$ and $B(2, 4)$ (Figure 4.12).

Now try Exercise 1.

EXAMPLE 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval [–1, 1].

(a) $f(x) = \sqrt{x^2} + 1$ **(b)** $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases}$

SOLUTION

(a) Note that $\sqrt{x^2} + 1 = |x| + 1$, so this is just a vertical shift of the absolute value function, which has a nondifferentiable "corner" at $x = 0$. (See Section 3.2.) The function f is not differentiable on (–1, 1).

(b) Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 + 3 = 4$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 1 = 2$, the function has a discontinuity at $x = 1$. The function f is not continuous on [–1, 1].

If the two functions given had satisfied the necessary conditions, the *conclusion* of the Mean Value Theorem would have guaranteed the existence of a number c in (– 1, 1) such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = 0$. Such a number c does not exist for the function in part (a), but one happens to exist for the function in part (b) (Figure 4.13).

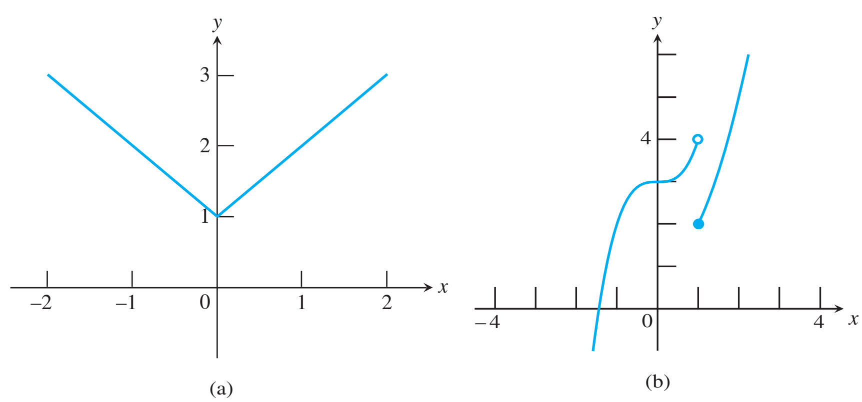


Figure 4.13 For both functions in Example 2, $\dfrac{f(1) - f(-1)}{1 - (-1)} = 0$ but neither function satisfies the conditions of the Mean Value Theorem on the interval [– 1, 1]. For the function in Example 2(a), there is no number c such that $f'(c) = 0$. It happens that $f'(0) = 0$ in Example 2(b).

Now try Exercise 3.

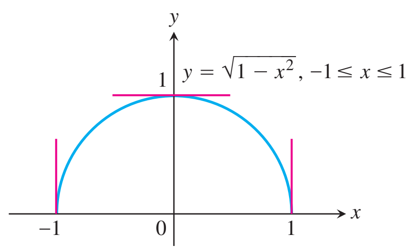


Figure 4.14 (Example 3)

EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

SOLUTION

The function f (Figure 4.14) is continuous on the interval $[-1, 1]$ and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. We find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent $y = 1$. ***Now try Exercise 9.***

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

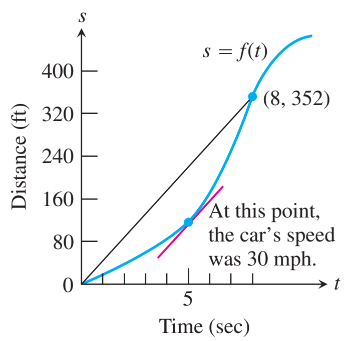


Figure 4.15 (Example 4)

EXAMPLE 4 Interpreting the Mean Value Theorem

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 4.15).

Now try Exercise 11.

Increasing and Decreasing Functions

Our first use of the Mean Value Theorem will be its application to increasing and decreasing functions.

Monotonic Functions

A function that is always increasing on an interval or always decreasing on an interval is said to be **monotonic** there.

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

The Mean Value Theorem allows us to identify exactly where graphs rise and fall. Functions with positive derivatives are increasing functions; functions with negative derivatives are decreasing functions.

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

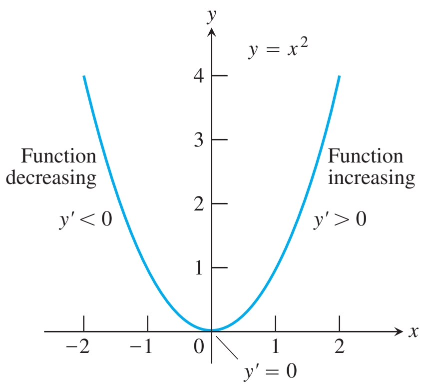
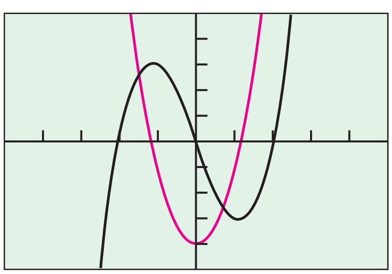


Figure 4.16 (Example 5)

What's Happening at Zero?

Note that 0 appears in both intervals in Example 5, which is consistent both with the definition and with Corollary 1. Does this mean that the function $y = x^2$ is both increasing and decreasing at $x = 0$? No! This is because a function can only be described as increasing or decreasing on an *interval* with more than one point (see the definition). Saying that $y = x^2$ is "increasing at $x = 2$" is not really proper either, but you will often see that statement used as a short way of saying $y = x^2$ is "increasing on an interval containing 2."



[–5, 5] by [–5, 5]

Figure 4.17 By comparing the graphs of $f(x) = x^3 - 4x$ and $f'(x) = 3x^2 - 4$ we can relate the increasing and decreasing behavior of f to the sign of f'. (Example 6)

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore,

(a) $f(x_1) < f(x_2)$ if $f' > 0$ on (a, b) (f is increasing), or

(b) $f(x_1) > f(x_2)$ if $f' < 0$ on (a, b) (f is decreasing). ■

EXAMPLE 5 Determining Where Graphs Rise or Fall

The function $y = x^2$ (Figure 4.16) is

(a) decreasing on $(-\infty, 0]$ because $y' = 2x < 0$ on $(-\infty, 0)$.

(b) increasing on $[0, \infty)$ because $y' = 2x > 0$ on $(0, \infty)$. ***Now try Exercise 15.***

EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreasing?

SOLUTION

Solve Graphically The graph of f in Figure 4.17 suggests that f is increasing from $-\infty$ to the x-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the x-coordinate of the local minimum to ∞. This information is supported by the superimposed graph of $f'(x) = 3x^2 - 4$.

Confirm Analytically The function is increasing where $f'(x) > 0$.

$$3x^2 - 4 > 0$$
$$x^2 > \frac{4}{3}$$
$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

The function is decreasing where $f'(x) < 0$.

$$3x^2 - 4 < 0$$
$$x^2 < \frac{4}{3}$$
$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

In interval notation, f is increasing on $(-\infty, -\sqrt{4/3}]$, decreasing on $[-\sqrt{4/3}, \sqrt{4/3}]$, and increasing on $[\sqrt{4/3}, \infty)$. ***Now try Exercise 27.***

Other Consequences

We know that constant functions have the zero function as their derivative. We can now use the Mean Value Theorem to show conversely that the only functions with the zero function as derivative are constant functions.

COROLLARY 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

Proof Our plan is to show that $f(x_1) = f(x_2)$ for any two points x_1 and x_2 in I. We can assume the points are numbered so that $x_1 < x_2$. Since f is differentiable at every point of $[x_1, x_2]$, it is continuous at every point as well. Thus, f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$. Therefore, there is a point c between x_1 and x_2 for which

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) = 0$, it follows that $f(x_1) = f(x_2)$. ■

We can use Corollary 2 to show that if two functions have the same derivative, they differ by a constant.

COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof Let $h = f - g$. Then for each point x in I,

$$h'(x) = f'(x) - g'(x) = 0.$$

It follows from Corollary 2 that there is a constant C such that $h(x) = C$ for all x in I. Thus, $h(x) = f(x) - g(x) = C$, or $f(x) = g(x) + C$. ■

We know that the derivative of $f(x) = x^2$ is $2x$ on the interval $(-\infty, \infty)$. So, any other function $g(x)$ with derivative $2x$ on $(-\infty, \infty)$ must have the formula $g(x) = x^2 + C$ for some constant C.

EXAMPLE 7 Applying Corollary 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

SOLUTION

Since f has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$, for some constant C. To identify C, we use the condition that the graph must pass through (0, 2). This is equivalent to saying that

$$\begin{aligned} f(0) &= 2 \\ -\cos(0) + C &= 2 \qquad f(x) = -\cos x + C \\ -1 + C &= 2 \\ C &= 3. \end{aligned}$$

The formula for f is $f(x) = -\cos x + 3$. *Now try Exercise 35.*

In Example 7 we were given a derivative and asked to find a function with that derivative. This type of function is so important that it has a name.

DEFINITION Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

We know that if f has one antiderivative F then it has infinitely many antiderivatives, each differing from F by a constant. Corollary 3 says these are all there are. In Example 7, we found the particular antiderivative of $\sin x$ whose graph passed through the point $(0, 2)$.

EXAMPLE 8 Finding Velocity and Position

Find the velocity and position functions of a body falling freely from a height of 0 meters under each of the following sets of conditions:

(a) The acceleration is $9.8\ \mathrm{m/sec^2}$ and the body falls from rest.

(b) The acceleration is $9.8\ \mathrm{m/sec^2}$ and the body is propelled downward with an initial velocity of $1\ \mathrm{m/sec}$.

SOLUTION

(a) ***Falling from rest.*** We measure distance fallen in meters and time in seconds, and assume that the body is released from rest at time $t = 0$.

Velocity: We know that the velocity $v(t)$ is an antiderivative of the constant function 9.8. We also know that $g(t) = 9.8t$ is an antiderivative of 9.8. By Corollary 3,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus,

$$9.8(0) + C = 0 \quad \text{and} \quad C = 0.$$

The body's velocity function is $v(t) = 9.8t$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t$. We also know that $h(t) = 4.9t^2$ is an antiderivative of $9.8t$. By Corollary 3,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2$.

(b) ***Propelled downward.*** We measure distance fallen in meters and time in seconds, and assume that the body is propelled downward with velocity of $1\ \mathrm{m/sec}$ at time $t = 0$.

Velocity: The velocity function still has the form $9.8t + C$, but instead of being zero, the initial velocity (velocity at $t = 0$) is now $1\ \mathrm{m/sec}$. Thus,

$$9.8(0) + C = 1 \quad \text{and} \quad C = 1.$$

The body's velocity function is $v(t) = 9.8t + 1$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t + 1$. We also know that $k(t) = 4.9t^2 + t$ is an antiderivative of $9.8t + 1$. By Corollary 3,

$$s(t) = 4.9t^2 + t + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + 0 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2 + t$. ***Now try Exercise 43.***

Quick Review 4.2 *(For help, go to Sections 1.2, 2.3, and 3.2.)*

In Exercises 1 and 2, find exact solutions to the inequality.

1. $2x^2 - 6 < 0$ $(-\sqrt{3}, \sqrt{3})$ **2.** $3x^2 - 6 > 0$ $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

In Exercises 3–5, let $f(x) = \sqrt{8 - 2x^2}$.

3. Find the domain of f. $[-2, 2]$

4. Where is f continuous? For all x in its domain, or, $[-2, 2]$

5. Where is f differentiable? On $(-2, 2)$

In Exercises 6–8, let $f(x) = \dfrac{x}{x^2 - 1}$.

6. Find the domain of f. $x \neq \pm 1$

7. Where is f continuous? For all x in its domain, or, for all $x \neq \pm 1$

8. Where is f differentiable? For all x in its domain, or, for all $x \neq \pm 1$

In Exercises 9 and 10, find C so that the graph of the function f passes through the specified point.

9. $f(x) = -2x + C, \quad (-2, 7)$ $C = 3$

10. $g(x) = x^2 + 2x + C, \quad (1, -1)$ $C = -4$

Section 4.2 Exercises

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

2. $f(x) = x^{2/3}$ on $[0, 1]$

3. $f(x) = x^{1/3}$ on $[-1, 1]$ No. There is a vertical tangent at $x = 0$.

4. $f(x) = |x - 1|$ on $[0, 4]$ No. There is a corner at $x = 1$.

5. $f(x) = \sin^{-1} x$ on $[-1, 1]$

6. $f(x) = \ln(x - 1)$ on $[2, 4]$

7. $f(x) = \begin{cases} \cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$ on $[0, \pi]$

No. The split function is discontinuous at $x = \dfrac{\pi}{2}$.

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

No. The split function is discontinuous at $x = 1$.

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line AB.

(b) a tangent line to f in the interval (a, b) that is parallel to AB.

9. $f(x) = x + \dfrac{1}{x}$, $0.5 \le x \le 2$ **(a)** $y = \dfrac{5}{2}$ **(b)** $y = 2$

10. $f(x) = \sqrt{x - 1}$, $1 \le x \le 3$ See page 204.

11. ***Speeding*** A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

12. ***Temperature Change*** It took 20 sec for the temperature to rise from 0°F to 212°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.

13. ***Triremes*** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

14. ***Running a Marathon*** A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

15. $f(x) = 5x - x^2$ See page 204. **16.** $g(x) = x^2 - x - 12$ See page 204.

17. $h(x) = \dfrac{2}{x}$ See page 204. **18.** $k(x) = \dfrac{1}{x^2}$ See page 204.

19. $f(x) = e^{2x}$ See page 204. **20.** $f(x) = e^{-0.5x}$ See page 204.

21. $y = 4 - \sqrt{x + 2}$ See page 204. **22.** $y = x^4 - 10x^2 + 9$ See page 204.

1. (a) Yes. **(b)** $2c + 2 = \dfrac{2 - (-1)}{1 - 0} = 3$, so $c = \dfrac{1}{2}$.

2. (a) Yes. **(b)** $\dfrac{2}{3}c^{-1/3} = \dfrac{1 - 0}{1 - 0} = 1$, so $c = \dfrac{8}{27}$.

5. (a) Yes. **(b)** $\dfrac{1}{\sqrt{1 - c^2}} = \dfrac{(\pi/2) - (-\pi/2)}{1 - (-1)} = \dfrac{\pi}{2}$, so $c = \sqrt{1 - 4/\pi^2} \approx 0.771$.

6. (a) Yes. **(b)** $\dfrac{1}{c - 1} = \dfrac{\ln 3 - \ln 1}{4 - 2}$, so $c \approx 2.820$.

23. (a) Local max at $\approx (2.67, 3.08)$; local min at $(4, 0)$
(b) On $(-\infty, 8/3]$ **(c)** On $[8/3, 4]$

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

24. (a) Local min at $\approx (-2, -7.56)$ **(b)** On $[-2, \infty)$ **(c)** On $(-\infty, -2]$

23. $f(x) = x\sqrt{4 - x}$

24. $g(x) = x^{1/3}(x + 8)$

25. $h(x) = \dfrac{-x}{x^2 + 4}$

26. $k(x) = \dfrac{x}{x^2 - 4}$

27. $f(x) = x^3 - 2x - 2\cos x$

28. $g(x) = 2x + \cos x$
(a) None **(b)** On $(-\infty, \infty)$ **(c)** None

In Exercises 29–34, find all possible functions f with the given derivative.

29. $f'(x) = x$ $\quad \dfrac{x^2}{2} + C$

30. $f'(x) = 2$ $\quad 2x + C$

31. $f'(x) = 3x^2 - 2x + 1$ $\quad x^3 - x^2 + x + C$

32. $f'(x) = \sin x$ $\quad -\cos x + C$

33. $f'(x) = e^x$ $\quad e^x + C$

34. $f'(x) = \dfrac{1}{x - 1}, \quad x > 1$ $\quad \ln(x - 1) + C$

In Exercises 35–38, find the function with the given derivative whose graph passes through the point P.

35. $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$ $\quad \dfrac{1}{x} + \dfrac{1}{2}, x > 0$

36. $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$ $\quad x^{1/4} - 3$

37. $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$ $\quad \ln(x + 2) + 3$

38. $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$ $\quad x^2 + x - \sin x + 3$

Group Activity In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

39. **(a)** local minimum at $(1, 1)$, local maximum at $(3, 3)$

(b) local minima at $(1, 1)$ and $(3, 3)$

(c) local maxima at $(1, 1)$ and $(3, 3)$

40. $f(2) = 3, \quad f'(2) = 0,$ and

(a) $f'(x) > 0$ for $x < 2, \quad f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2, \quad f'(x) > 0$ for $x > 2$.

(c) $f'(x) < 0$ for $x \neq 2$.

(d) $f'(x) > 0$ for $x \neq 2$.

41. $f'(-1) = f'(1) = 0, \quad f'(x) > 0$ on $(-1, 1)$,
$f'(x) < 0$ for $x < -1, \quad f'(x) > 0$ for $x > 1$.

42. A local minimum value that is greater than one of its local maximum values.

43. ***Free Fall*** On the moon, the acceleration due to gravity is 1.6 m/sec^2.

(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later? 48 m/sec

(b) How far below the point of release is the bottom of the crevasse? 720 meters

(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does? After about 27.604 seconds, and it will be going about 48.166 m/sec

44. ***Diving*** **(a)** With what velocity will you hit the water if you step off from a 10-m diving platform? 14 m/sec

(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec? $10\sqrt{2}$ m/sec, or, about 14.142 m/sec

45. Writing to Learn The function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so f' is never zero between 0 and 1, and the graph of f has no tangent parallel to the chord from $(0, 0)$ to $(1, 0)$. Explain why this does not contradict the Mean Value Theorem. Because the function is not continuous on $[0, 1]$.

46. Writing to Learn Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

47. ***Unique Solution*** Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

In Exercises 48 and 49, show that the equation has exactly one solution in the interval. [*Hint:* See Exercise 47.]

48. $x^4 + 3x + 1 = 0, \quad -2 \leq x \leq -1$

49. $x + \ln(x + 1) = 0, \quad 0 \leq x \leq 3$

50. ***Parallel Tangents*** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. True or False If f is differentiable and increasing on (a, b), then $f'(c) > 0$ for every c in (a, b). Justify your answer.

52. True or False If f is differentiable and $f'(c) > 0$ for every c in (a, b), then f is increasing on (a, b). Justify your answer.

51. False. For example, the function x^3 is increasing on $(-1, 1)$, but $f'(0) = 0$.
52. True. In fact, f is increasing on $[a, b]$ by Corollary 1 to the Mean Value Theorem.

53. Multiple Choice If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$ A

(A) $-\frac{3}{2\pi}$ **(B)** $-\frac{\sqrt{3}}{2}$ **(C)** $-\frac{1}{2}$ **(D)** 0 **(E)** $\frac{1}{2}$

54. Multiple Choice On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing? B

(A) $(-\infty, 2]$ **(B)** $[0, 4]$ **(C)** $[2, 4]$ **(D)** $(4, \infty)$ **(E)** no interval

55. Multiple Choice Which of the following functions is an antiderivative of $\frac{1}{\sqrt{x}}$? E

(A) $-\frac{1}{\sqrt{2x^3}}$ **(B)** $-\frac{2}{\sqrt{x}}$ **(C)** $\frac{\sqrt{x}}{2}$ **(D)** $\sqrt{x} + 5$ **(E)** $2\sqrt{x} - 10$

56. Multiple Choice All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except* D

(A) $\sin x$ **(B)** $\sin^{-1} x$ **(C)** $x^{5/3}$ **(D)** $x^{3/5}$ **(E)** $\frac{x}{x-2}$

Explorations

57. *Analyzing Derivative Data* Assume that f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

x	$f'(x)$	x	$f'(x)$
−2	7	0.25	−4.81
−1.75	4.19	0.5	−4.25
−1.5	1.75	0.75	−3.31
−1.25	−0.31	1	−2
−1	−2	1.25	−0.31
−0.75	−3.31	1.5	1.75
−0.5	−4.25	1.75	4.19
−0.25	−4.81	2	7
0	−5		

(a) Estimate where f is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for f' and find a formula for f that satisfies $f(0) = 0$.

58. *Analyzing Motion Data* Priya's distance D in meters from a motion detector is given by the data in Table 4.1.

Table 4.1 Motion Detector Data

t (sec)	D (m)	t (sec)	D (m)
0.0	3.36	4.5	3.59
0.5	2.61	5.0	4.15
1.0	1.86	5.5	3.99
1.5	1.27	6.0	3.37
2.0	0.91	6.5	2.58
2.5	1.14	7.0	1.93
3.0	1.69	7.5	1.25
3.5	2.37	8.0	0.67
4.0	3.01		

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) Writing to Learn Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation $D = f(t)$ for the data in Table 4.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for f to find a formula for f'. Use this formula to estimate the answers to (a).

Extending the Ideas

59. *Geometric Mean* The **geometric mean** of two positive numbers a and b is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of c in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

60. *Arithmetic Mean* The **arithmetic mean** of two numbers a and b is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of c in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

61. *Upper Bounds* Show that for any numbers a and b, $|\sin b - \sin a| \le |b - a|$.

62. *Sign of f′* Assume that f is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

63. *Monotonic Functions* Show that monotonic increasing and decreasing functions are one-to-one.

Answers:

10. (a) $y = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}$, or $y \approx 0.707x - 0.707$

(b) $y = \frac{1}{\sqrt{2}}x - \frac{1}{2\sqrt{2}}$, or $y \approx 0.707x - 0.354$

15. (a) Local maximum at $\left(\frac{5}{2}, \frac{25}{4}\right)$ **(b)** On $\left(-\infty, \frac{5}{2}\right]$ **(c)** On $\left[\frac{5}{2}, \infty\right)$

16. (a) Local minimum at $\left(\frac{1}{2}, -\frac{49}{4}\right)$ **(b)** On $\left[\frac{1}{2}, \infty\right)$ **(c)** On $\left(-\infty, \frac{1}{2}\right]$

17. (a) None **(b)** None **(c)** On $(-\infty, 0)$ and $(0, \infty)$

18. (a) None **(b)** On $(-\infty, 0)$ **(c)** On $(0, \infty)$

19. (a) None **(b)** On $(-\infty, \infty)$ **(c)** None

20. (a) None **(b)** None **(c)** On $(-\infty, \infty)$

21. (a) Local maximum at $(-2, 4)$ **(b)** None **(c)** On $[-2, \infty)$

22. (a) Local maximum at $(0, 9)$; local minima at $(-\sqrt{5}, -16)$ and $(\sqrt{5}, -16)$ **(b)** On $[-\sqrt{5}, 0]$ and $[\sqrt{5}, \infty)$ **(c)** On $(-\infty, -\sqrt{5}]$ and $[0, \sqrt{5}]$

4.3 Connecting f' and f'' with the Graph of f

What you'll learn about

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

. . . and why

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

First Derivative Test for Local Extrema

As we see once again in Figure 4.18, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in a critical point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

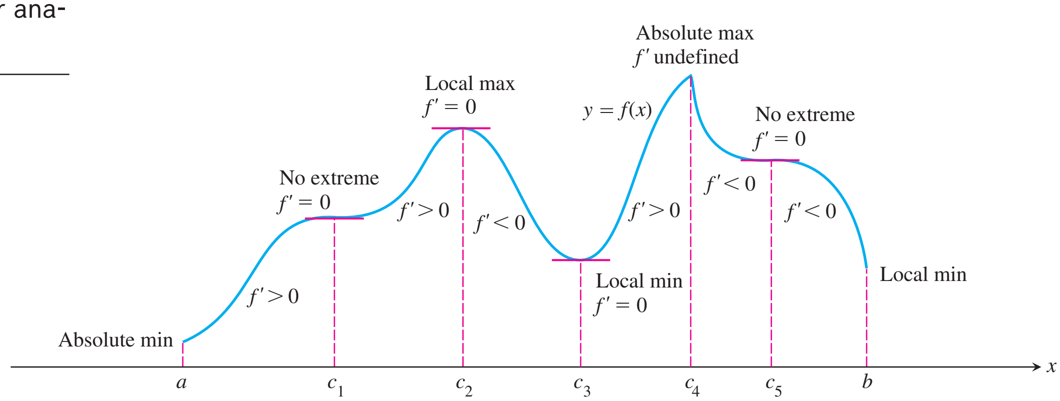


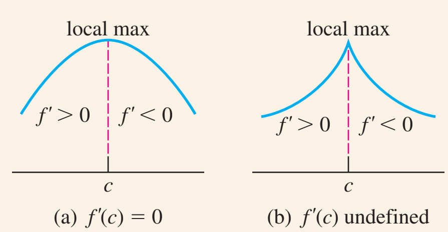
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

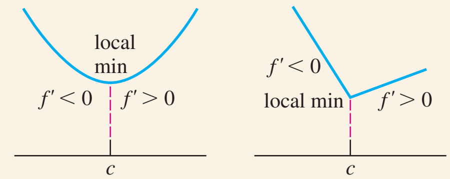
At a critical point c:

1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



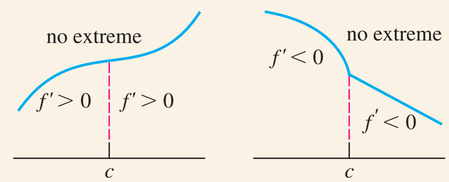
continued

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

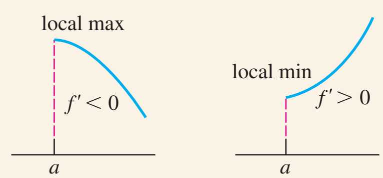
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

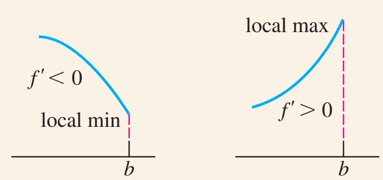
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



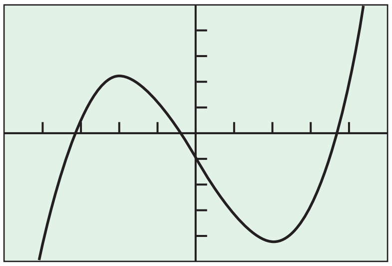
Here is how we apply the First Derivative Test to find the local extrema of a function. The critical points of a function f partition the x-axis into intervals on which f' is either positive or negative. We determine the sign of f' in each interval by evaluating f' for one value of x in the interval. Then we apply Theorem 4 as shown in Examples 1 and 2.

EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

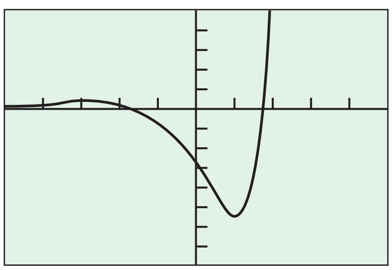
(a) $f(x) = x^3 - 12x - 5$ **(b)** $g(x) = (x^2 - 3)e^x$

continued



[–5, 5] by [–25, 25]

Figure 4.19 The graph of $f(x) = x^3 - 12x - 5$.

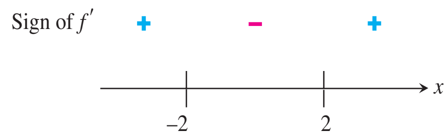


[–5, 5] by [–8, 5]

Figure 4.20 The graph of $g(x) = (x^2 - 3)e^x$.

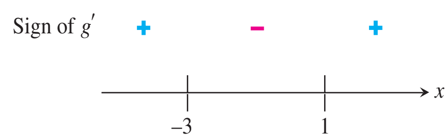
SOLUTION

(a) Since f is differentiable for all real numbers, the only possible critical points are the zeros of f'. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the x-axis into three intervals, as shown below:



Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The local maximum value is $f(-2) = 11$, and the local minimum value is $f(2) = -21$. There are no absolute extrema, as the function has range $(-\infty, \infty)$ (Figure 4.19).

(b) Since g is differentiable for all real numbers, the only possible critical points are the zeros of g'. Since $g'(x) = (x^2 - 3) \cdot e^x + (2x) \cdot e^x = (x^2 + 2x - 3) \cdot e^x$, we find the zeros of g' to be $x = 1$ and $x = -3$. The zeros partition the x-axis into three intervals, as shown below:



Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -3$ and a local minimum at $x = 1$. The local maximum value is $g(-3) = 6e^{-3} \approx 0.299$, and the local minimum value is $g(1) = -2e \approx -5.437$. Although this function has the same increasing–decreasing–increasing pattern as f, its left end behavior is quite different. We see that $\lim_{x\to-\infty} g(x) = 0$, so the graph approaches the y-axis asymptotically and is therefore bounded below. This makes $g(1)$ an *absolute* minimum. Since $\lim_{x\to\infty} g(x) = \infty$, there is no absolute maximum (Figure 4.20).

Now try Exercise 3.

Concavity

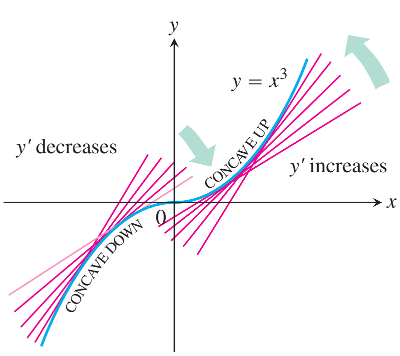


Figure 4.21 The graph of $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

As you can see in Figure 4.21, the function $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways. Looking at tangents as we scan from left to right, we see that the slope y' of the curve decreases on the interval $(-\infty, 0)$ and then increases on the interval $(0, \infty)$. The curve $y = x^3$ is *concave down* on $(-\infty, 0)$ and *concave up* on $(0, \infty)$. The curve lies below the tangents where it is concave down, and above the tangents where it is concave up.

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.

If a function $y = f(x)$ has a second derivative, then we can conclude that y' increases if $y'' > 0$ and y' decreases if $y'' < 0$.

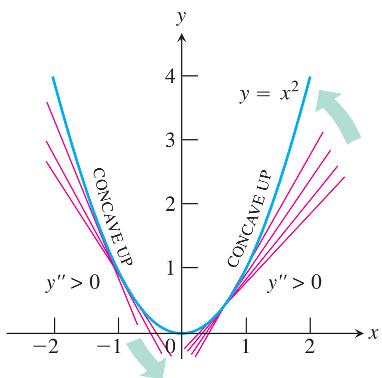
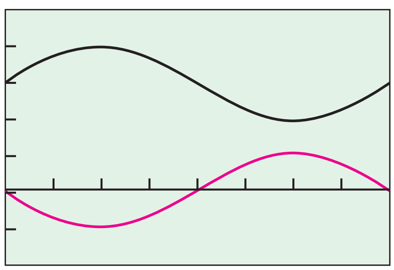


Figure 4.22 The graph of $y = x^2$ is concave up on any interval. (Example 2)



$[0, 2\pi]$ by $[-2, 5]$

Figure 4.23 Using the graph of y'' to determine the concavity of y. (Example 2)

Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

SOLUTION

(a) Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on *any* interval. In particular, it is concave up on $(3, 10)$ (Figure 4.22).

(b) The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.23).

Now try Exercise 7.

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined. If y is a twice differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

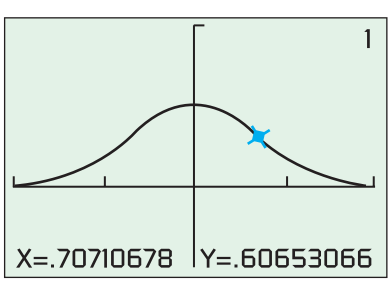
SOLUTION

First we find the second derivative, recalling the Chain and Product Rules:

$$\begin{aligned} y &= e^{-x^2} \\ y' &= e^{-x^2} \cdot (-2x) \\ y'' &= e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2) \\ &= e^{-x^2}(4x^2 - 2) \end{aligned}$$

The factor e^{-x^2} is always positive, while the factor $(4x^2 - 2)$ changes sign at $-\sqrt{1/2}$ and at $\sqrt{1/2}$. Since y'' must also change sign at these two numbers, the points of inflection are $(-\sqrt{1/2}, 1/\sqrt{e})$ and $(\sqrt{1/2}, 1/\sqrt{e})$. We confirm our solution graphically by observing the changes of curvature in Figure 4.24.

Now try Exercise 13.



$[-2, 2]$ by $[-1, 2]$

Figure 4.24 Graphical confirmation that the graph of $y = e^{-x^2}$ has a point of inflection at $x = \sqrt{1/2}$ (and hence also at $x = -\sqrt{1/2}$). (Example 3)

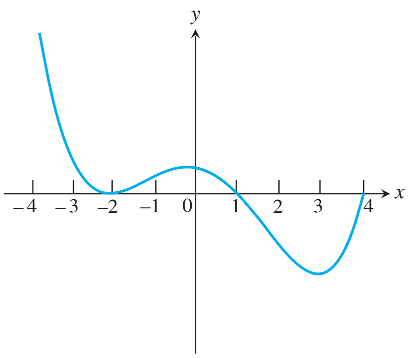


Figure 4.25 The graph of f', the derivative of f, on the interval $[-4, 4]$.

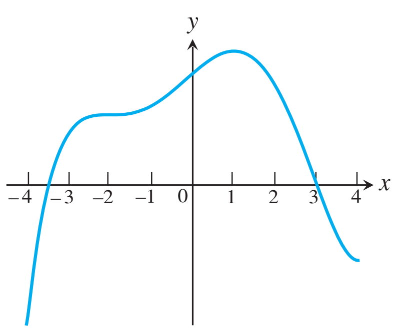
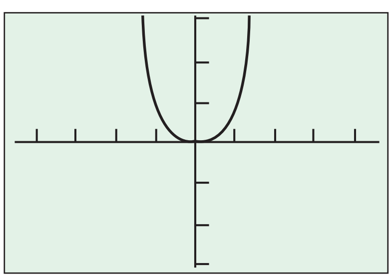
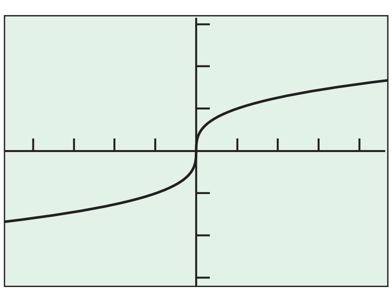


Figure 4.26 A possible graph of f. (Example 4)



[– 4.7, 4.7] by [–3.1, 3.1]

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[–4.7, 4.7] by [–3.1, 3.1]

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f, justifying each answer with information obtained from the graph of f'.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At which x-coordinates does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

(e) Sketch a possible graph of f on the interval $[-4, 4]$.

SOLUTION

(a) Since $f' > 0$ on the intervals $[-4, -2)$ and $(-2, 1)$, the function f must be increasing on the entire interval $[-4, 1]$ with a horizontal tangent at $x = -2$ (a "shelf point").

(b) The graph of f is concave up on the intervals where f' is increasing. We see from the graph that f' is increasing on the intervals $(-2, 0)$ and $(3, 4)$.

(c) By the First Derivative Test, there is a local maximum at $x = 1$ because the sign of f' changes from positive to negative there. Note that there is no extremum at $x = -2$, since f' does not change sign. Because the function increases from the left endpoint and decreases to the right endpoint, there are local minima at the endpoints $x = -4$ and $x = 4$.

(d) The inflection points of the graph of f have the same x-coordinates as the turning points of the graph of f', namely -2, 0, and 3.

(e) A possible graph satisfying all the conditions is shown in Figure 4.26.

Now try Exercise 23.

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

1. *The second derivative can be zero at a noninflection point*. For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point*. For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points of inflection.

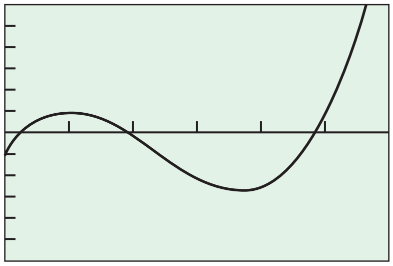
EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x-axis with position function

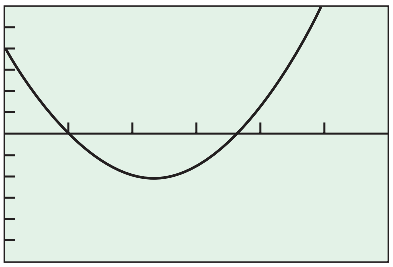
$$x(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

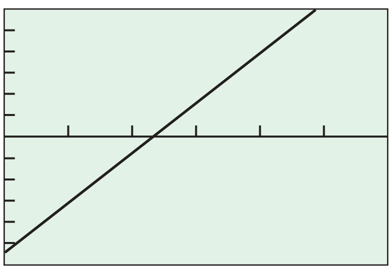
continued



[0, 6] by [–30, 30]
(a)



[0, 6] by [–30, 30]
(b)



[0, 6] by [–30, 30]
(c)

Figure 4.29 The graph of (a) $x(t) = 2t^3 - 14t^2 + 22t - 5$, $t \geq 0$, (b) $x'(t) = 6t^2 - 28t + 22$, and (c) $x''(t) = 12t - 28$. (Example 5)

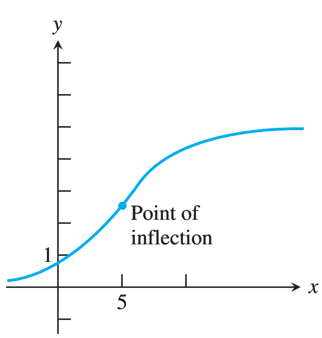


Figure 4.30 A logistic curve

$$y = \frac{c}{1 + ae^{-bx}}.$$

SOLUTION

Solve Analytically

The velocity is

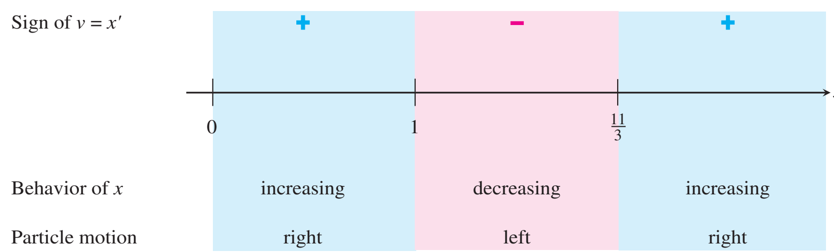
$$v(t) = x'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = x''(t) = 12t - 28 = 4(3t - 7).$$

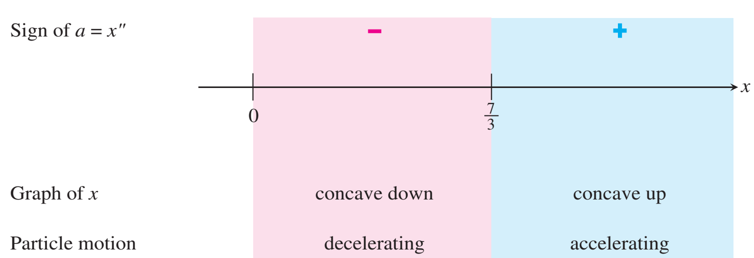
When the function $x(t)$ is increasing, the particle is moving to the right on the x-axis; when $x(t)$ is decreasing, the particle is moving to the left. Figure 4.29 shows the graphs of the position, velocity, and acceleration of the particle.

Notice that the first derivative ($v = x'$) is zero when $t = 1$ and $t = 11/3$. These zeros partition the t-axis into three intervals, as shown in the sign graph of v below:



The particle is moving to the right in the time intervals [0, 1) and (11/3, ∞) and moving to the left in (1, 11/3).

The acceleration $a(t) = 12t - 28$ has a single zero at $t = 7/3$. The sign graph of the acceleration is shown below:



The accelerating force is directed toward the left during the time interval [0, 7/3], is momentarily zero at $t = 7/3$, and is directed toward the right thereafter.

Now try Exercise 25.

The growth of an individual company, of a population, in sales of a new product, or of salaries often follows a *logistic* or *life cycle curve* like the one shown in Figure 4.30. For example, sales of a new product will generally grow slowly at first, then experience a period of rapid growth. Eventually, sales growth slows down again. The function f in Figure 4.30 is increasing. Its rate of increase, f', is at first increasing ($f'' > 0$) up to the point of inflection, and then its rate of increase, f', is decreasing ($f'' < 0$). This is, in a sense, the opposite of what happens in Figure 4.21.

Some graphers have the logistic curve as a built-in regression model. We use this feature in Example 6.

Table 4.2 Population of Alaska

Years since 1900	Population
20	55,036
30	59,278
40	75,524
50	128,643
60	226,167
70	302,583
80	401,851
90	550,043
100	626,932

Source: Bureau of the Census, U.S. Chamber of Commerce.

EXAMPLE 6 Population Growth in Alaska

Table 4.2 shows the population of Alaska in each 10-year census between 1920 and 2000.

(a) Find the logistic regression for the data.

(b) Use the regression equation to predict the Alaskan population in the 2020 census.

(c) Find the inflection point of the regression equation. What significance does the inflection point have in terms of population growth in Alaska?

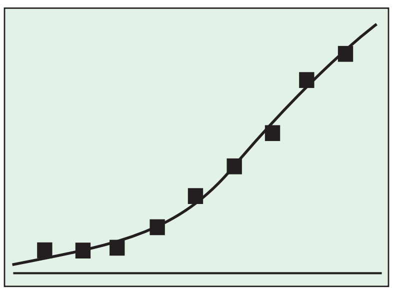
(d) What does the regression equation indicate about the population of Alaska in the long run?

SOLUTION

(a) Using years since 1900 as the independent variable and population as the dependent variable, the logistic regression equation is approximately

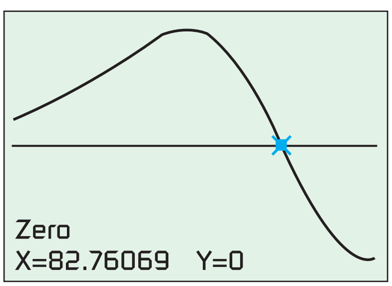
$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}.$$

Its graph is superimposed on a scatter plot of the data in Figure 4.31(a). Store the regression equation as Y1 in your calculator.



[12, 108] by [0, 730000]

(a)



[12, 108] by [–250, 250]

(b)

Figure 4.31 (a) The logistic regression curve

$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}$$

superimposed on the population data from Table 4.2, and (b) the graph of y'' showing a zero at about $x = 83$.

(b) The calculator reports Y1(120) to be approximately 781,253. (Given the uncertainty of this kind of extrapolation, it is probably more reasonable to say "approximately 781,200.")

(c) The inflection point will occur where y'' changes sign. Finding y'' algebraically would be tedious, but we can graph the numerical derivative of the numerical derivative and find the zero graphically. Figure 4.31(b) shows the graph of y'', which is nDeriv(nDeriv(Y1,X,X),X,X) in calculator syntax. The zero is approximately 83, so the inflection point occurred in 1983, when the population was about 450,570 and growing the fastest.

(d) Notice that $\lim_{x\to\infty} \dfrac{895598}{1+ 71.57e^{-0.0516x}} = 895598$, so the regression equation indicates that the population of Alaska will stabilize at about 895,600 in the long run. Do not put too much faith in this number, however, as human population is dependent on too many variables that can, and will, change over time. ***Now try Exercise 31.***

Second Derivative Test for Local Extrema

Instead of looking for sign changes in y' at critical points, we can sometimes use the following test to determine the presence of local extrema.

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test fails if $f''(c) = 0$ or if $f''(c)$ fails to exist. When this happens, go back to the First Derivative Test for local extreme values.

In Example 7, we apply the Second Derivative Test to the function in Example 1.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

SOLUTION

We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$f''(x) = 6x.$$

Testing the critical points $x = \pm 2$ (there are no endpoints), we find

$$f''(-2) = -12 < 0 \Rightarrow f \text{ has a local maximum at } x = -2 \text{ and}$$

$$f''(2) = 12 > 0 \Rightarrow f \text{ has a local minimum at } x = 2.$$

Now try Exercise 35.

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch a possible graph for f.

SOLUTION

f is continuous since f' exists. The domain of f' is $(-\infty, \infty)$, so the domain of f is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test and the table above we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above we see that f is decreasing in $(-\infty, 0]$ and $[0, 3]$, and increasing in $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

continued

Note

The *Second Derivative Test* does not apply at $x = 0$ because $f''(0) = 0$. We need the First Derivative Test to see that there is no local extremum at $x = 0$.

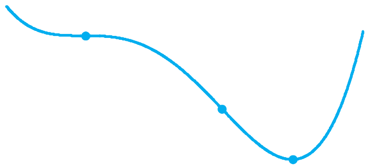


Figure 4.32 The graph for f has no extremum but has points of inflection where $x = 0$ and $x = 2$, and a local minimum where $x = 3$. (Example 8)

(d) Summarizing the information in the two tables above we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$x < 3$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

Figure 4.32 shows one possibility for the graph of f. ***Now try Exercise 39.***

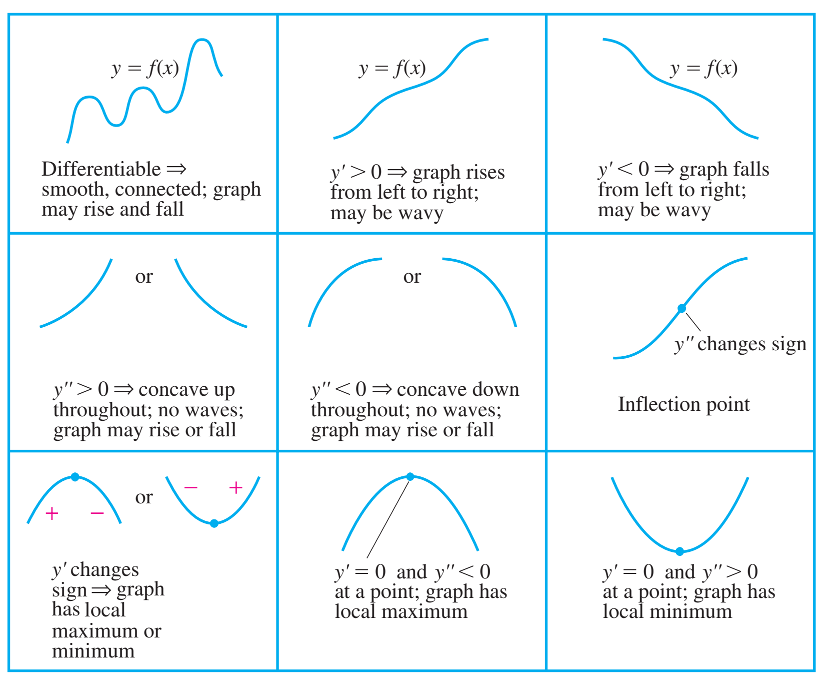
EXPLORATION 1 **Finding f from f'**

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives

We have seen in Example 8 and Exploration 1 that we are able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where any local extrema are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the xy-plane. As we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point.



Remember also that a function can be continuous and still have points of nondifferentiability (cusps, corners, and points with vertical tangent lines). Thus, a noncontinuous graph of f' could lead to a continuous graph of f, as Example 9 shows.

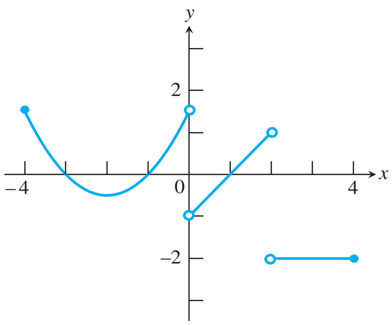


Figure 4.33 The graph of f', a discontinuous derivative.

EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f', with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x-coordinates of all local extrema and points of inflection of f.

(b) Sketch a possible graph of f.

SOLUTION

(a) For extrema, we look for places where f' changes sign. There are local maxima at $x = -3$, 0, and 2 (where f' goes from positive to negative) and local minima at $x = -1$ and 1 (where f' goes from negative to positive). There are also local minima at the two endpoints $x = -4$ and 4, because f' starts positive at the left endpoint and ends negative at the right endpoint.

For points of inflection, we look for places where f'' changes sign, that is, where the graph of f' changes direction. This occurs only at $x = -2$.

(b) A possible graph of f is shown in Figure 4.34. The derivative information determines the shape of the three components, and the continuity condition determines that the three components must be linked together. ***Now try Exercises 49 and 53.***

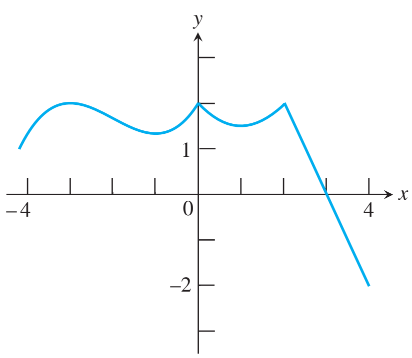


Figure 4.34 A possible graph of f. (Example 9)

EXPLORATION 2 Finding f from f' and f''

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	$+$	does not exist	$-$	0	$-$
f''	$+$	does not exist	$+$	0	$-$

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.

Quick Review 4.3 *(For help, go to Sections 1.3, 2.2, 3.3, and 3.9.)*

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

1. $x^2 - 9 < 0$ $(-3, 3)$

2. $x^3 - 4x > 0$ $(-2, 0) \cup (2, \infty)$

In Exercises 3–6, find the domains of f and f'.

3. $f(x) = xe^x$ f: all reals; f': all reals

4. $f(x) = x^{3/5}$ f: all reals; f': $x \neq 0$

5. $f(x) = \dfrac{x}{x-2}$ f: $x \neq 2$; f': $x \neq 2$

6. $f(x) = x^{2/5}$ f: all reals; f': $x \neq 0$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

7. $y = (4 - x^2)e^x$ $y = 0$

8. $y = (x^2 - x)e^{-x}$ $y = 0$

9. $y = \dfrac{200}{1 + 10e^{-0.5x}}$ $y = 0$ and $y = 200$

10. $y = \dfrac{750}{2 + 5e^{-0.1x}}$ $y = 0$ and $y = 375$

1. Local minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$; $-\frac{5}{4}$ is an absolute minimum. **2.** Local maximum: (2, 5); local minimum: (0, −3)

Section 4.3 Exercises

3. Local maximum: (0, 1); local minima: (−1, −1) and (1, −1); −1 is an absolute minimum.

In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

1. $y = x^2 - x - 1$

2. $y = -2x^3 + 6x^2 - 3$

3. $y = 2x^4 - 4x^2 + 1$

4. $y = xe^{1/x}$ Local minimum: $(1, e)$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$; local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$; 4 is an absolute maximum and −4 is an absolute minimum.

5. $y = x\sqrt{8 - x^2}$

Local minimum: (0, 1)

6. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 7–12, use the Concavity Test to determine the intervals on which the graph of the function is **(a)** concave up and **(b)** concave down.

7. $y = 4x^3 + 21x^2 + 36x - 20$
(a) $(-7/4, \infty)$ **(b)** $(-\infty, -7/4)$

8. $y = -x^4 + 4x^3 - 4x + 1$
(a) $(0, 2)$ **(b)** $(-\infty, 0)$ and $(2, \infty)$

9. $y = 2x^{1/5} + 3$
(a) $(-\infty, 0)$ **(b)** $(0, \infty)$

10. $y = 5 - x^{1/3}$
(a) $(0, \infty)$ **(b)** $(-\infty, 0)$

11. $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \geq 1 \end{cases}$
(a) None **(b)** $(1, \infty)$

12. $y = e^x, \quad 0 \leq x \leq 2\pi$
(a) $(0, 2\pi)$ **(b)** None

In Exercises 13–20, find all points of inflection of the function.

13. $y = xe^x$ $(-2, -2/e^2)$

14. $y = x\sqrt{9 - x^2}$ $(0, 0)$

15. $y = \tan^{-1} x$ $(0, 0)$

16. $y = x^3(4 - x)$ $(0, 0)$ and $(2, 16)$

17. $y = x^{1/3}(x - 4)$
$(0, 0)$ and $(-2, 6\sqrt[3]{2})$

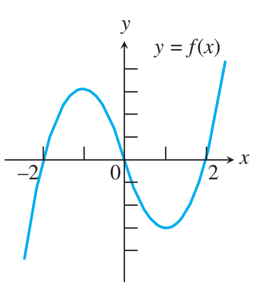
18. $y = x^{1/2}(x + 3)$ $(1, 4)$

19. $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$
$(1, 1)$

20. $y = \dfrac{x}{x^2 + 1}$ $(0, 0)$, $(\sqrt{3}, \sqrt{3}/4)$, and $(-\sqrt{3}, -\sqrt{3}/4)$

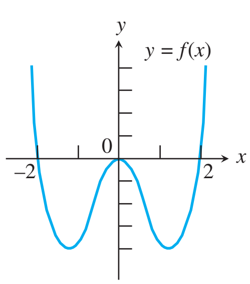
In Exercises 21 and 22, use the graph of the function f to estimate where **(a)** f' and **(b)** f'' are 0, positive, and negative.

21.



(a) Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$
(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

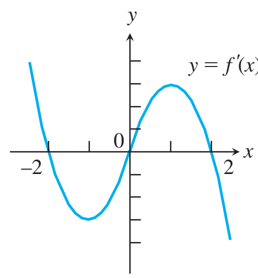
22.



(a) Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$
(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

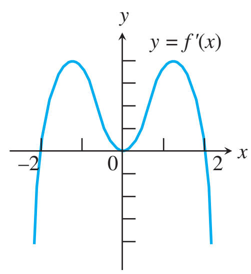
In Exercises 23 and 24, use the graph of the function f' to estimate the intervals on which the function f is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the x-coordinates of all local extreme values.

23.



(a) $(-\infty, -2]$ and $[0, 2]$
(b) $[-2, 0]$ and $[2, \infty)$
(c) Local maxima: $x = -2$ and $x = 2$; local minimum: $x = 0$

24.



(a) $[-2, 2]$ **(b)** $(-\infty, -2]$ and $[2, \infty)$
(c) Local maximum: $x = 2$;
local minimum: $x = -2$

In Exercises 25–28, a particle is moving along the x-axis with position function $x(t)$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \geq 0$.

25. $x(t) = t^2 - 4t + 3$

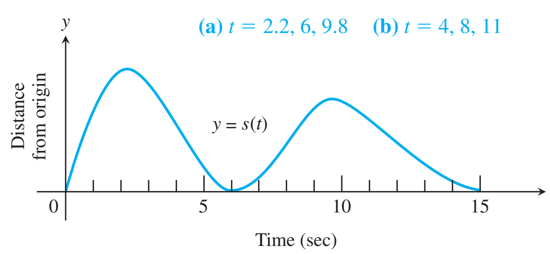
26. $x(t) = 6 - 2t - t^2$

27. $x(t) = t^3 - 3t + 3$

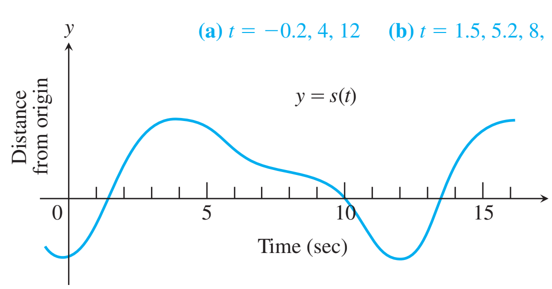
28. $x(t) = 3t^2 - 2t^3$

In Exercises 29 and 30, the graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's **(a)** velocity equal to zero? **(b)** acceleration equal to zero?

29. **(a)** $t = 2.2, 6, 9.8$ **(b)** $t = 4, 8, 11$



30. **(a)** $t = -0.2, 4, 12$ **(b)** $t = 1.5, 5.2, 8, 11, 13$



31. Table 4.3 shows the population of Pennsylvania in each 10-year census between 1830 and 1950.

Table 4.3 Population of Pennsylvania

Years since 1820	Population in thousands
10	1348
20	1724
30	2312
40	2906
50	3522
60	4283
70	5258
80	6302
90	7665
100	8720
110	9631
120	9900
130	10,498

Source: Bureau of the Census, U.S. Chamber of Commerce.

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Pennsylvania population in the 2000 census.

(d) In what year was the Pennsylvania population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Pennsylvania in the long run?

32. In 1977, there were 12,168,450 basic cable television subscribers in the U.S. Table 4.4 shows the cumulative number of subscribers added to that baseline number from 1978 to 1985.

Table 4.4 Growth of Cable Television

Years since 1977	Added Subscribers since 1977
1	1,391,910
2	2,814,380
3	5,671,490
4	11,219,200
5	17,340,570
6	22,113,790
7	25,290,870
8	27,872,520

Source: Nielsen Media Research, as reported in *The World Almanac and Book of Facts 2004.*

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve. Does it fit the data well?

(c) In what year between 1977 and 1985 were basic cable TV subscriptions growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(d) What does the regression equation indicate about the number of basic cable television subscribers in the long run? (Be sure to add the baseline 1977 number.)

(e) Writing to Learn In fact, the long-run number of basic cable subscribers predicted by the regression equation falls short of the actual 2002 number by more than 32 million. What circumstances changed to render the earlier model so ineffective?

In Exercises 33–38, use the Second Derivative Test to find the local extrema for the function.

33. $y = 3x - x^3 + 5$

34. $y = x^5 - 80x + 100$

35. $y = x^3 + 3x^2 - 2$

36. $y = 3x^5 - 25x^3 + 60x + 20$

37. $y = xe^x$

38. $y = xe^{-x}$

In Exercises 39 and 40, use the derivative of the function $y = f(x)$ to find the points at which f has a

(a) local maximum, **(b)** local minimum, or

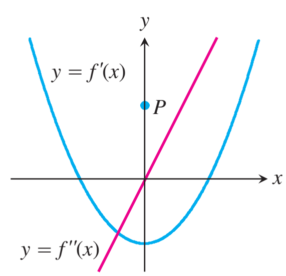
(c) point of inflection.

39. $y' = (x-1)^2(x-2)$ **(a)** None **(b)** At $x = 2$ **(c)** At $x = 1$ and $x = \frac{5}{3}$

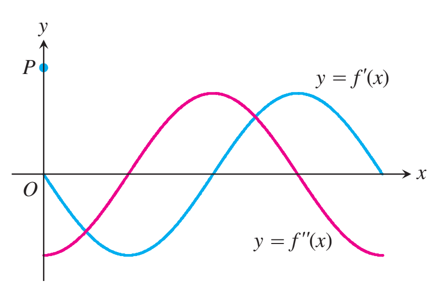
40. $y' = (x-1)^2(x-2)(x-4)$
(a) At $x = 2$ **(b)** At $x = 4$ **(c)** At $x = 1$, $x \approx 1.63$, $x \approx 3.37$

Exercises 41 and 42 show the graphs of the first and second derivatives of a function $y = f(x)$. Copy the figure and add a sketch of a possible graph of f that passes through the point P.

41.



42.



43. **Writing to Learn** If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point c of f's domain, must f have a local maximum or minimum at $x = c$? Explain.

44. **Writing to Learn** If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point c of f's domain, must f have an inflection point at $x = c$? Explain.

45. ***Connecting f and f′*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

46. ***Connecting f and f″*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

47. ***Connecting f, f′, and f″*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

$f(-2) = 8$	$f'(x) > 0$ for $\lvert x\rvert > 2$
$f(0) = 4$	$f'(x) < 0$ for $\lvert x\rvert < 2$
$f(2) = 0$	$f''(x) < 0$ for $x < 0$
$f'(2) = f'(-2) = 0$	$f''(x) > 0$ for $x > 0$

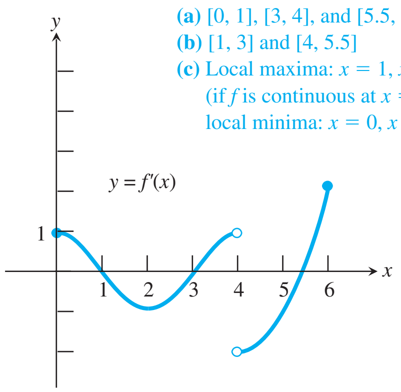
48. ***Using Behavior to Sketch*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Curve
$x < 2$		falling, concave up
2	1	horizontal tangent
$2 < x < 4$		rising, concave up
4	4	inflection point
$4 < x < 6$		rising, concave down
6	7	horizontal tangent
$x > 6$		falling, concave down

In Exercises 49 and 50, use the graph of f' to estimate the intervals on which the function f is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the x-coordinates of all local extreme values. (Assume that the function f is continuous, even at the points where f' is undefined.)

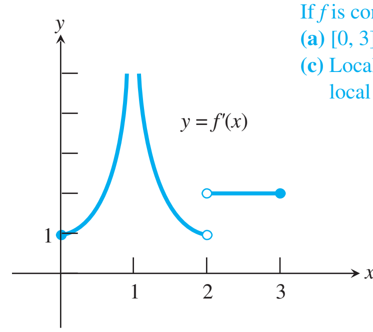
49. The domain of f' is $[0, 4) \cup (4, 6]$.

(a) $[0, 1]$, $[3, 4]$, and $[5.5, 6]$
(b) $[1, 3]$ and $[4, 5.5]$
(c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$; local minima: $x = 0$, $x = 3$, and $x = 5.5$



50. The domain of f' is $[0, 1) \cup (1, 2) \cup (2, 3]$.

If f is continuous on $[0, 3]$:
(a) $[0, 3]$ **(b)** Nowhere
(c) Local maximum: $x = 3$; local minimum: $x = 0$



Group Activity In Exercises 51 and 52, do the following.

(a) Find the absolute extrema of f and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of f.

51. f is continuous on $[0, 3]$ and satisfies the following.

x	0	1	2	3
f	0	2	0	-2
f'	3	0	does not exist	-3
f''	0	-1	does not exist	0

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	+	+	−
f'	+	−	−
f''	−	−	−

52. f is an even function, continuous on $[-3, 3]$, and satisfies the following.

x	0	1	2
f	2	0	-1
f'	does not exist	0	does not exist
f''	does not exist	0	does not exist

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	+	−	−
f'	−	−	+
f''	+	−	−

(d) What can you conclude about $f(3)$ and $f(-3)$?

Group Activity In Exercises 53 and 54, sketch a possible graph of a continuous function f that has the given properties.

53. Domain $[0, 6]$, graph of f' given in Exercise 49, and $f(0) = 2$.

54. Domain $[0, 3]$, graph of f' given in Exercise 50, and $f(0) = -3$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

55. True or False If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection. Justify your answer.
False. For example, consider $f(x) = x^4$ at $c = 0$.

56. True or False If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Justify your answer.
True. This is the Second Derivative Test for a local maximum.

57. Multiple Choice If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on A

(A) $\left(-\infty, -\dfrac{1}{a}\right)$ **(B)** $\left(-\infty, \dfrac{1}{a}\right)$ **(C)** $\left(-\dfrac{1}{a}, \infty\right)$

(D) $\left(\dfrac{1}{a}, \infty\right)$ **(E)** $(-\infty, -1)$

58. Multiple Choice If $f(0) = f'(0) = f''(0) = 0$, which of the following *must be true*? E

(A) There is a local maximum of f at the origin.
(B) There is a local minimum of f at the origin.
(C) There is no local extremum of f at the origin.
(D) There is a point of inflection of the graph of f at the origin.
(E) There is a horizontal tangent to the graph of f at the origin.

59. Multiple Choice The x-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are C

(A) 0 only **(B)** 1 only **(C)** 3 only **(D)** 0 and 3 **(E)** 0 and 1

60. Multiple Choice Which of the following conditions would enable you to conclude that the graph of f has a point of inflection at $x = c$? A

(A) There is a local maximum of f' at $x = c$.
(B) $f''(c) = 0$.
(C) $f''(c)$ does not exist.
(D) The sign of f' changes at $x = c$.
(E) f is a cubic polynomial and $c = 0$.

Exploration

61. *Graphs of Cubics* There is almost no leeway in the locations of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of x. Check this out by examining **(a)** the cubic in Exercise 7 and **(b)** the cubic in Exercise 2. Then **(c)** prove the general case.

Extending the Ideas

In Exercises 62 and 63, feel free to use a CAS (computer algebra system), if you have one, to solve the problem.

62. *Logistic Functions* Let $f(x) = c/(1 + ae^{-bx})$ with $a > 0$, $abc \neq 0$.

(a) Show that f is increasing on the interval $(-\infty, \infty)$ if $abc > 0$, and decreasing if $abc < 0$.

(b) Show that the point of inflection of f occurs at $x = (\ln |a|)/b$.

63. *Quartic Polynomial Functions* Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

(a) Show that the graph of f has 0 or 2 points of inflection.

(b) Write a condition that must be satisfied by the coefficients if the graph of f has 0 or 2 points of inflection.

Quick Quiz for AP* Preparation: Sections 4.1–4.3

You should solve these problems without using a graphing calculator.

1. Multiple Choice How many critical points does the function $f(x) = (x - 2)^5 (x + 3)^4$ have? C

(A) One **(B)** Two **(C)** Three **(D)** Five **(E)** Nine

2. Multiple Choice For what value of x does the function $f(x) = (x - 2)(x - 3)^2$ have a relative maximum? D

(A) -3 **(B)** $-\dfrac{7}{3}$ **(C)** $-\dfrac{5}{2}$ **(D)** $\dfrac{7}{3}$ **(E)** $\dfrac{5}{2}$

3. Multiple Choice If g is a differentiable function such that $g(x) < 0$ for all real numbers x, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true? B

(A) f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

(B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.

(C) f has relative minima at $x = -3$ and at $x = 3$.

(D) f has relative maxima at $x = -3$ and at $x = 3$.

(E) It cannot be determined if f has any relative extrema.

4. Free Response Let f be the function given by $f(x) = 3 \ln (x^2 + 2) - 2x$ with domain $[-2, 4]$.

(a) Find the coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f.

(c) Find the absolute maximum value of $f(x)$.

4.4 Modeling and Optimization

What you'll learn about

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena with Differentiable Functions

. . . and why

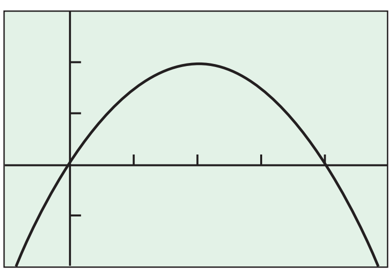
Historically, optimization problems were among the earliest applications of what we now call differential calculus.

Examples from Mathematics

While today's graphing technology makes it easy to find extrema without calculus, the algebraic methods of differentiation were understandably more practical, and certainly more accurate, when graphs had to be rendered by hand. Indeed, one of the oldest applications of what we now call "differential calculus" (pre-dating Newton and Leibniz) was to find maximum and minimum values of functions by finding where horizontal tangent lines might occur. We will use both algebraic and graphical methods in this section to solve "max-min" problems in a variety of contexts, but the emphasis will be on the *modeling* process that both methods have in common. Here is a strategy you can use:

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.



[–5, 25] by [–100, 150]

Figure 4.35 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

SOLUTION

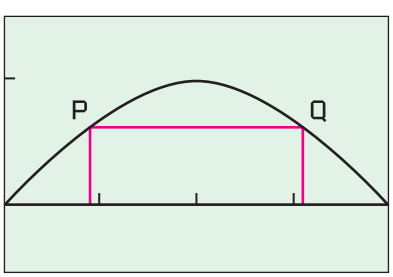
Model If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

Solve Graphically We can see from the graph of f in Figure 4.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Interpret The two numbers we seek are $x = 10$ and $20 - x = 10$.

Now try Exercise 1.

Sometimes we find it helpful to use both analytic and graphical methods together, as in Example 2.



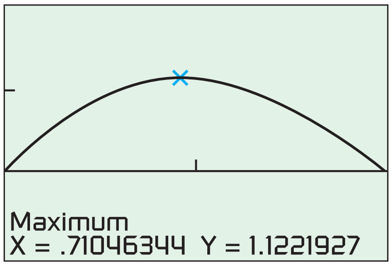
[0, π] by [–0.5, 1.5]

Figure 4.36 A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

EXAMPLE 2 Inscribing Rectangles

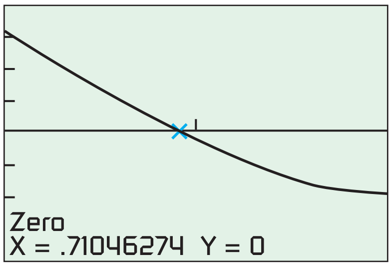
A rectangle is to be inscribed under one arch of the sine curve (Figure 4.36). What is the largest area the rectangle can have, and what dimensions give that area?

continued



[0, $\pi/2$] by [–1, 2]

(a)



[0, $\pi/2$] by [–4, 4]

(b)

Figure 4.37 The graph of (a) $A(x) = (\pi - 2x)\sin x$ and (b) A' in the interval $0 \le x \le \pi/2$. (Example 2)

SOLUTION

Model Let $(x, \sin x)$ be the coordinates of point P in Figure 4.36. From what we know about the sine function the x-coordinate of point Q is $(\pi - x)$. Thus,

$$\pi - 2x = \text{length of rectangle}$$

and

$$\sin x = \text{height of rectangle.}$$

The area of the rectangle is

$$A(x) = (\pi - 2x)\sin x.$$

Solve Analytically and Graphically We can assume that $0 \le x \le \pi/2$. Notice that $A = 0$ at the endpoints $x = 0$ and $x = \pi/2$. Since A is differentiable, the only critical points occur at the zeros of the first derivative,

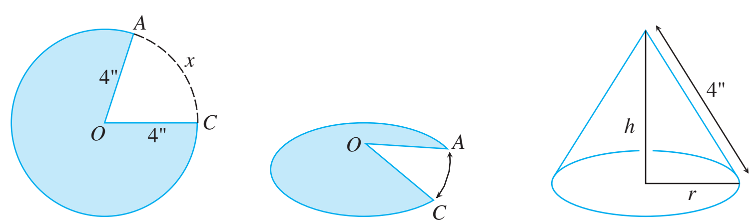
$$A'(x) = -2\sin x + (\pi - 2x)\cos x.$$

It is not possible to solve the equation $A'(x) = 0$ using algebraic methods. We can use the graph of A (Figure 4.37a) to find the maximum value and where it occurs. Or, we can use the graph of A' (Figure 4.37b) to find where the derivative is zero, and then evaluate A at this value of x to find the maximum value. The two x-values appear to be the same, as they should.

Interpret The rectangle has a maximum area of about 1.12 square units when $x \approx 0.71$. At this point, the rectangle is $\pi - 2x \approx 1.72$ units long by $\sin x \approx 0.65$ unit high. *Now try Exercise 5.*

EXPLORATION 1 Constructing Cones

A cone of height h and radius r is constructed from a flat, circular disk of radius 4 in. by removing a sector AOC of arc length x in. and then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is that volume?



NOT TO SCALE

1. Show that

$$r = \frac{8\pi - x}{2\pi}, \quad h = \sqrt{16 - r^2}, \quad \text{and}$$

$$V(x) = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}.$$

2. Show that the natural domain of V is $0 \le x \le 16\pi$. Graph V over this domain.
3. Explain why the restriction $0 \le x \le 8\pi$ makes sense in the problem situation. Graph V over this domain.
4. Use graphical methods to find where the cone has its maximum volume, and what that volume is.
5. Confirm your findings in part 4 analytically. [*Hint:* Use $V(x) = (1/3)\pi r^2 h$, $h^2 + r^2 = 16$, and the Chain Rule.]

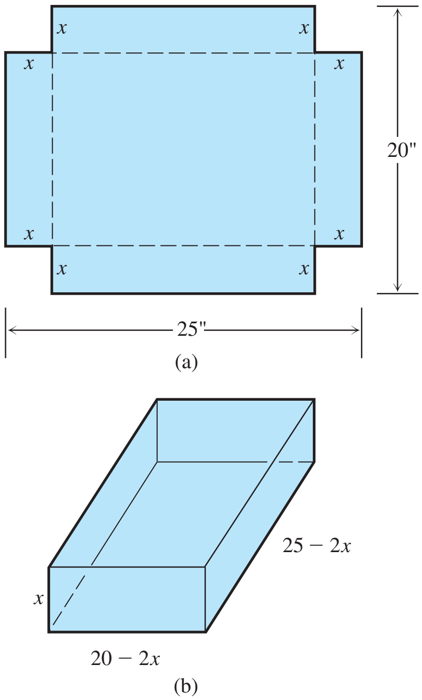
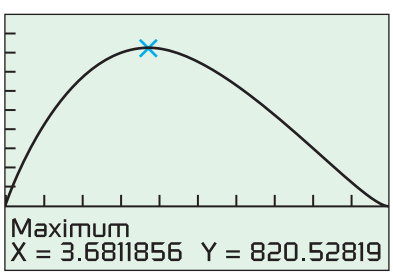


Figure 4.38 An open box made by cutting the corners from a piece of tin. (Example 3)

$y = x(20 - 2x)(25 - 2x)$



[0, 10] by [–300, 1000]

Figure 4.39 We chose the -300 in $-300 \le y \le 1000$ so that the coordinates of the local maximum at the bottom of the screen would not interfere with the graph. (Example 3)

Examples from Business and Industry

To *optimize* something means to maximize or minimize some aspect of it. What is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest rectangular beam we can cut from a 12-inch log? We usually answer such questions by finding the greatest or smallest value of some function that we have used to model the situation.

EXAMPLE 3 Fabricating a Box

An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20- by 25-inch sheet of tin and bending up the sides (Figure 4.38). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

SOLUTION

Model The height of the box is x, and the other two dimensions are $(20 - 2x)$ and $(25 - 2x)$. Thus, the volume of the box is

$$V(x) = x(20 - 2x)(25 - 2x).$$

Solve Graphically Because $2x$ cannot exceed 20, we have $0 \le x \le 10$. Figure 4.39 suggests that the maximum value of V is about 820.53 and occurs at $x \approx 3.68$.

Confirm Analytically Expanding, we obtain $V(x) = 4x^3 - 90x^2 + 500x$. The first derivative of V is

$$V'(x) = 12x^2 - 180x + 500.$$

The two solutions of the quadratic equation $V'(x) = 0$ are

$$c_1 = \frac{180 - \sqrt{180^2 - 48(500)}}{24} \approx 3.68 \qquad \text{and}$$

$$c_2 = \frac{180 + \sqrt{180^2 - 48(500)}}{24} \approx 11.32.$$

Only c_1 is in the domain $[0, 10]$ of V. The values of V at this one critical point and the two endpoints are

Critical point value: $V(c_1) \approx 820.53$

Endpoint values: $V(0) = 0, \quad V(10) = 0.$

Interpret Cutout squares that are about 3.68 in. on a side give the maximum volume, about 820.53 in^3. ***Now try Exercise 7.***

EXAMPLE 4 Designing a Can

You have been asked to design a one-liter oil can shaped like a right circular cylinder (see Figure 4.40 on the next page). What dimensions will use the least material?

continued

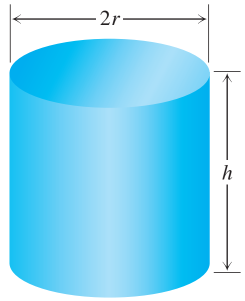
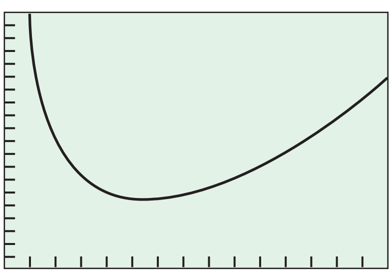


Figure 4.40 This one-liter can uses the least material when $h = 2r$. (Example 4)



[0, 15] by [0, 2000]

Figure 4.41 The graph of $A = 2\pi r^2 + 2000/r$, $r > 0$. (Example 4)

SOLUTION

Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad \text{1 liter = 1000 cm}^3$$

$$\textit{Surface area of can:} \quad A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi rh}_{\text{cylinder wall}}$$

How can we interpret the phrase "least material"? One possibility is to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 17 describes one way to take waste into account.)

Model To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier,

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Solve Analytically Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.41 suggests that such a value exists.

Notice from the graph that for small r (a tall thin container, like a piece of pipe), the term $2000/r$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\begin{aligned} \frac{dA}{dr} &= 4\pi r - \frac{2000}{r^2} \\ 0 &= 4\pi r - \frac{2000}{r^2} && \text{Set } dA/dr = 0. \\ 4\pi r^3 &= 2000 && \text{Multiply by } r^2. \\ r &= \sqrt[3]{\frac{500}{\pi}} \approx 5.42 && \text{Solve for } r. \end{aligned}$$

Something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is an open interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by referring to the shape of A's graph. The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore concave up and the value of A at $r = \sqrt[3]{500/\pi}$ an absolute minimum.

continued

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Interpret The one-liter can that uses the least material has height equal to the diameter, with $r \approx 5.42$ cm and $h \approx 10.84$ cm. *Now try Exercise 11.*

Examples from Economics

Here we want to point out two more places where calculus makes a contribution to economic theory. The first has to do with maximizing profit. The second has to do with minimizing average cost.

Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items,}$$

$$c(x) = \text{the cost of producing the } x \text{ items,}$$

$$p(x) = r(x) - c(x) = \text{the profit from selling } x \text{ items.}$$

The marginal revenue, marginal cost, and marginal profit at this production level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue,} \qquad \frac{dc}{dx} = \text{marginal cost,} \qquad \frac{dp}{dx} = \text{marginal profit.}$$

The first observation is about the relationship of p to these derivatives.

Marginal Analysis

Because differentiable functions are locally linear, we can use the marginals to approximate the extra revenue, cost, or profit resulting from selling or producing one more item. Using these approximations is referred to as *marginal analysis*.

THEOREM 6 Maximum Profit

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, so if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \quad \text{or} \quad r'(x) = c'(x).$$ ■

Figure 4.42 gives more information about this situation.

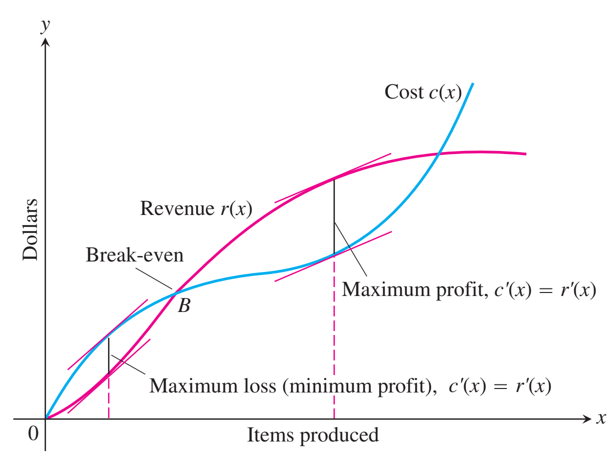


Figure 4.42 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, the maximum profit occurring where $r'(x) = c'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

What guidance do we get from this observation? We know that a production level at which $p'(x) = 0$ need not be a level of maximum profit. It might be a level of minimum profit, for example. But if we are making financial projections for our company, we should look for production levels at which marginal cost seems to equal marginal revenue. If there is a most profitable production level, it will be one of these.

EXAMPLE 5 Maximizing Profit

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

SOLUTION

Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \quad \text{Set } c'(x) = r'(x).$$

$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ thousand units or $x \approx 3.414$ thousand units. The graphs in Figure 4.43 show that maximum profit occurs at about $x = 3.414$ and maximum loss occurs at about $x = 0.586$.

Another way to look for optimal production levels is to look for levels that minimize the average cost of the units produced. Theorem 7 helps us find them. ***Now try Exercise 23.***

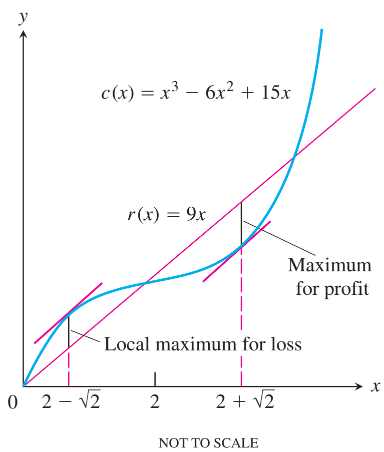


Figure 4.43 The cost and revenue curves for Example 5.

THEOREM 7 Minimizing Average Cost

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We assume that $c(x)$ is differentiable.

$$c(x) = \text{cost of producing } x \text{ items}, \ x > 0$$

$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items}$$

If the average cost can be minimized, it will be a production level at which

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0 \quad \text{Quotient Rule}$$

$$xc'(x) - c(x) = 0 \quad \text{Multiply by } x^2.$$

$$\underbrace{c'(x)}_{\text{marginal cost}} = \underbrace{\frac{c(x)}{x}}_{\text{average cost}}.$$

■

Again we have to be careful about what Theorem 7 does and does not say. It does not say that there is a production level of minimum average cost—it says where to look to see if there is one. Look for production levels at which average cost and marginal cost are equal. Then check to see if any of them gives a minimum average cost.

EXAMPLE 6 Minimizing Average Cost

Suppose $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that minimizes average cost? If so, what is it?

SOLUTION

We look for levels at which average cost equals marginal cost.

$$\text{Marginal cost:} \quad c'(x) = 3x^2 - 12x + 15$$

$$\text{Average cost:} \quad \frac{c(x)}{x} = x^2 - 6x + 15$$

$$\begin{aligned} 3x^2 - 12x + 15 &= x^2 - 6x + 15 \qquad \text{Marginal cost = Average cost} \\ 2x^2 - 6x &= 0 \\ 2x(x - 3) &= 0 \\ x = 0 \quad &\text{or} \quad x = 3 \end{aligned}$$

Since $x > 0$, the only production level that might minimize average cost is $x = 3$ thousand units.

We use the second derivative test.

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$$

The second derivative is positive for all $x > 0$, so $x = 3$ gives an absolute minimum. ***Now try Exercise 25.***

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ and $r(x)$ to describe the cost and revenue that comes from producing a number of items x that can only be an integer, here is the rationale.

When x is large, we can reasonably fit the cost and revenue data with smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x but at the values in between just as we do when we use regression equations. Once we have these differentiable functions, which are supposed to behave like the real cost and revenue when x is an integer, we can apply calculus to draw conclusions about their values. We then translate these mathematical conclusions into inferences about the real world that we hope will have predictive value. When they do, as is the case with the economic theory here, we say that the functions give a good model of reality.

What do we do when our calculus tells us that the best production level is a value of x that isn't an integer, as it did in Example 5? We use the nearest convenient integer. For $x \approx 3.414$ thousand units in Example 5, we might use 3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

Quick Review 4.4 *(For help, go to Sections 1.6, 4.1, and Appendix A.1.)*

1. Use the first derivative test to identify the local extrema of $y = x^3 - 6x^2 + 12x - 8$. None

2. Use the second derivative test to identify the local extrema of $y = 2x^3 + 3x^2 - 12x - 3$. Local maximum: $(-2, 17)$; local minimum: $(1, -10)$

3. Find the volume of a cone with radius 5 cm and height 8 cm. $\frac{200\pi}{3}$ cm^3

4. Find the dimensions of a right circular cylinder with volume $1000\ \text{cm}^3$ and surface area $600\ \text{cm}^2$. $r \approx 4.01$ cm and $h \approx 19.82$ cm, or, $r \approx 7.13$ cm and $h \approx 6.26$ cm

In Exercises 5–8, rewrite the expression as a trigonometric function of the angle α.

5. $\sin(-\alpha)$ $-\sin\alpha$

6. $\cos(-\alpha)$ $\cos\alpha$

7. $\sin(\pi - \alpha)$ $\sin\alpha$

8. $\cos(\pi - \alpha)$ $-\cos\alpha$

In Exercises 9 and 10, use substitution to find the exact solutions of the system of equations.

9. $\begin{cases} x^2 + y^2 = 4 \\ y = \sqrt{3}x \end{cases}$ $x = 1$ and $y = \sqrt{3}$, or, $x = -1$ and $y = -\sqrt{3}$

10. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ y = x + 3 \end{cases}$ $x = 0$ and $y = 3$, or, $x = -\frac{24}{13}$ and $y = \frac{15}{13}$

1. (a) As large as possible: 0 and 20; as small as possible: 10 and 10

(b) As large as possible: $\frac{79}{4}$ and $\frac{1}{4}$; as small as possible: 0 and 20

2. Largest area $= \frac{25}{4}$, dimensions are $\frac{5}{\sqrt{2}}$ cm by $\frac{5}{\sqrt{2}}$ cm

Section 4.4 Exercises

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

(a) the sum of their squares is as large as possible; as small as possible.

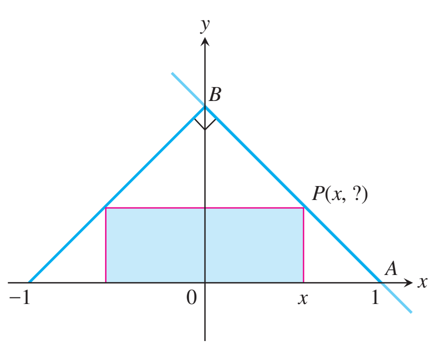
(b) one number plus the square root of the other is as large as possible; as small as possible.

2. ***Maximizing Area*** What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is $16\ \text{in}^2$, and what are its dimensions? Smallest perimeter = 16 in., dimensions are 4 in. by 4 in.

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square. See page 232.

5. ***Inscribing Rectangles*** The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



(a) Express the y-coordinate of P in terms of x. [*Hint:* Write an equation for the line AB.] $y = 1 - x$

(b) Express the area of the rectangle in terms of x. $A(x) = 2x(1 - x)$

(c) What is the largest area the rectangle can have, and what are its dimensions? Largest area $= \frac{1}{2}$, dimensions are 1 by $\frac{1}{2}$

6. ***Largest Rectangle*** A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions? Largest area = 32, dimensions are 4 by 8

7. ***Optimal Dimensions*** You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume? See page 232.

8. ***Closing Off the First Quadrant*** You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$. See page 232.

9. ***The Best Fencing Plan*** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions? Largest area = $80{,}000\ \text{m}^2$; dimensions: 200 m (perpendicular to river) by 400 m (parallel to river)

10. ***The Shortest Fence*** A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed? Dimensions: 12 m (divider is this length) by 18 m; total length required: 72 m

11. ***Designing a Tank*** Your iron works has contracted to design and build a 500-ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

(a) What dimensions do you tell the shop to use? 10 ft by 10 ft by 5 ft

(b) **Writing to Learn** Briefly describe how you took weight into account. Assume that the weight is minimized when the total area of the bottom and the 4 sides is minimized.

12. ***Catching Rainwater*** A 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.

(a) If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it? See page 232.

(b) **Writing to Learn** Give a possible scenario for the cost function in (a). See page 232.

13. ***Designing a Poster*** You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used? 18 in. high by 9 in. wide

14. ***Vertical Motion*** The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

(a) 96 ft/sec
(b) 256 feet at $t = 3$ seconds
(c) -128 ft/sec

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. ***Finding an Angle*** Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? [*Hint:* $A = (1/2)\,ab \sin\theta$.] $\theta = \frac{\pi}{2}$

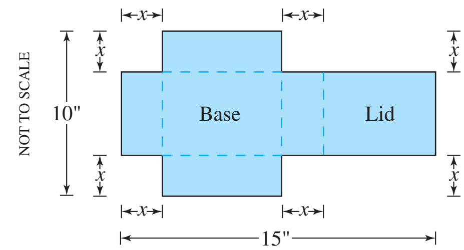
16. ***Designing a Can*** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 4. Radius = height = $10\pi^{-1/3}$ cm ≈ 6.83 cm. In Example 4, because of the top on the can, the "best" design is less big around and taller.

17. ***Designing a Can*** You are designing a 1000-cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 4. In Example 4 the ratio of h to r for the most economical can was 2 to 1. What is the ratio now? $\frac{8}{\pi}$ to 1

18. ***Designing a Box with Lid*** A piece of cardboard measures 10- by 15-in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



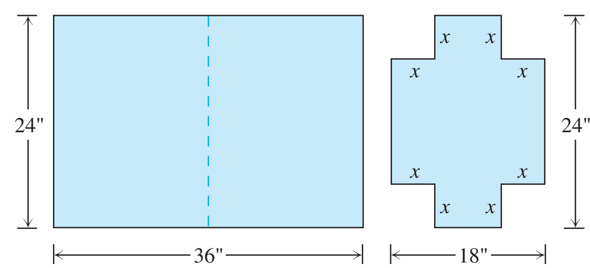
(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

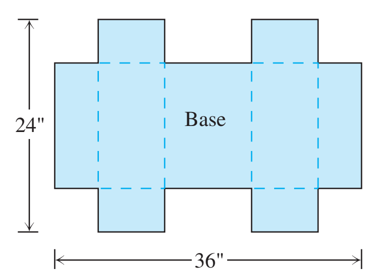
(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

19. ***Designing a Suitcase*** A 24- by 36-in. sheet of cardboard is folded in half to form a 24- by 18-in. rectangle as shown in the figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.



The sheet is then unfolded.



(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

(e) Find a value of x that yields a volume of 1120 in^3.

(f) **Writing to Learn** Write a paragraph describing the issues that arise in part (b).

20. $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

20. *Quickest Route* Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

21. *Inscribing Rectangles* A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area? Dimensions: width ≈ 3.44, height ≈ 2.61; maximum area ≈ 8.98

22. *Maximizing Volume* Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

23. *Maximizing Profit* Suppose $r(x) = 8\sqrt{x}$ represents revenue and $c(x) = 2x^2$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

24. *Maximizing Profit* Suppose $r(x) = x^2/(x^2 + 1)$ represents revenue and $c(x) = (x - 1)^3/3 - 1/3$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

25. *Minimizing Average Cost* Suppose $c(x) = x^3 - 10x^2 - 30x$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

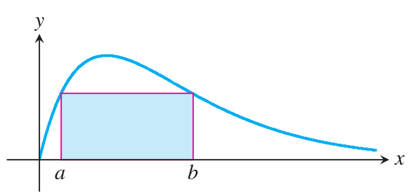
26. *Minimizing Average Cost* Suppose $c(x) = xe^x - 2x^2$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

27. *Tour Service* You operate a tour service that offers the following rates:

- \$200 per person if 50 people (the minimum number to book the tour) go on the tour.
- For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit? 67 people

28. *Group Activity* The figure shows the graph of $f(x) = xe^{-x}$, $x \geq 0$.



(a) Find where the absolute maximum of f occurs.

(b) Let $a > 0$ and $b > 0$ be given as shown in the figure. Complete the following table where A is the area of the rectangle in the figure.

a	b	A
0.1		
0.2		
0.3		
⋮		
1		

(c) Draw a scatter plot of the data (a, A).

(d) Find the quadratic, cubic, and quartic regression equations for the data in part (b), and superimpose their graphs on a scatter plot of the data.

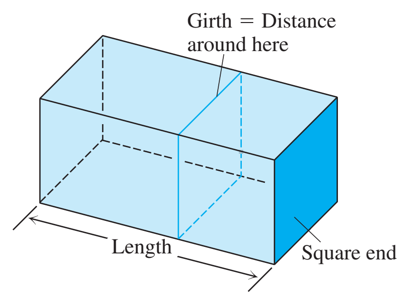
(e) Use each of the regression equations in part (d) to estimate the maximum possible value of the area of the rectangle.

29. *Cubic Polynomial Functions*
Let $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$.

(a) Show that f has either 0 or 2 local extrema. See page 232.

(b) Give an example of each possibility in part (a). See page 232.

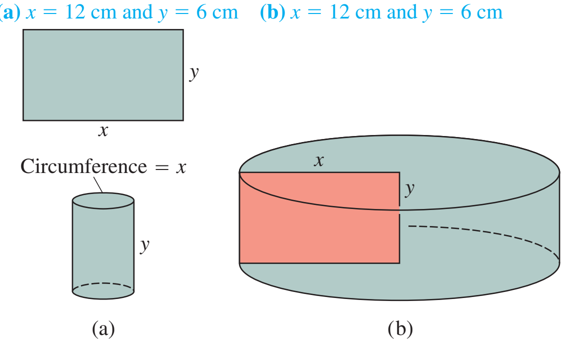
30. *Shipping Packages* The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around), as shown in the figure, does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume? 18 in. by 18 in. by 36 in.



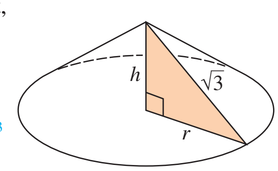
31. *Constructing Cylinders* Compare the answers to the following two construction problems.

(a) A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

(b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?
(a) $x = 12$ cm and $y = 6$ cm **(b)** $x = 12$ cm and $y = 6$ cm



32. *Constructing Cones* A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



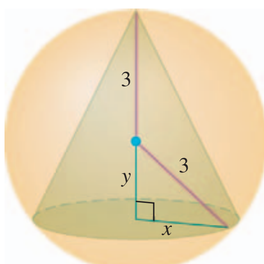
Radius $= \sqrt{2}$ m, height $= 1$ m, volume $\frac{2\pi}{3}$ m^3

33. *Finding Parameter Values* What value of a makes $f(x) = x^2 + (a/x)$ have **(a)** a local minimum at $x = 2$? **(b)** a point of inflection at $x = 1$? **(a)** $a = 16$ **(b)** $a = -1$

34. *Finding Parameter Values* Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a. See page 232.

35. *Finding Parameter Values* What values of a and b make $f(x) = x^3 + ax^2 + bx$ have **(a)** a local maximum at $x = -1$ and a local minimum at $x = 3$? **(b)** a local minimum at $x = 4$ and a point of inflection at $x = 1$? **(a)** $a = -3$ and $b = -9$ **(b)** $a = -3$ and $b = -24$

36. *Inscribing a Cone* Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. $\frac{32\pi}{3}$ cubic units

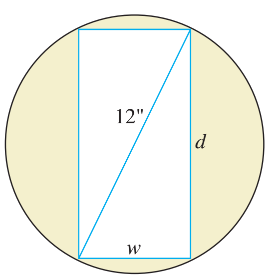


37. *Strength of a Beam* The strength S of a rectangular wooden beam is proportional to its width times the square of its depth.

(a) Find the dimensions of the strongest beam that can be cut from a 12-in. diameter cylindrical log.

(b) Writing to Learn Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) Writing to Learn On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.



38. *Stiffness of a Beam* The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

(a) Find the dimensions of the stiffest beam that can be cut from a 12-in. diameter cylindrical log.

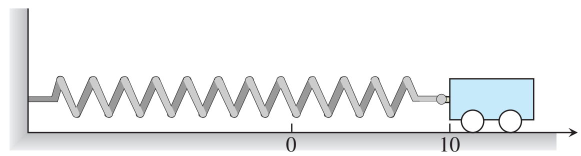
(b) Writing to Learn Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) Writing to Learn On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.

39. *Frictionless Cart* A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

(a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

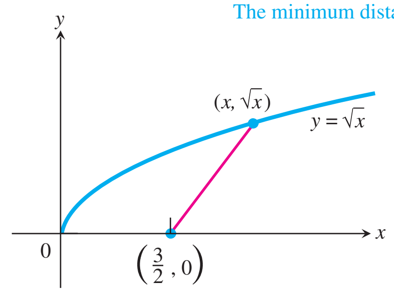
(b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



40. *Electrical Current* Suppose that at any time t (sec) the current i (amp) in an alternating current circuit is $i = 2 \cos t + 2 \sin t$. What is the peak (largest magnitude) current for this circuit? $2\sqrt{2}$ amps

41. *Calculus and Geometry* How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? [*Hint:* If you minimize the *square* of the distance, you can avoid square roots.]

The minimum distance is $\frac{\sqrt{5}}{2}$.



42. *Calculus and Geometry* How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$? The minimum distance is 2.

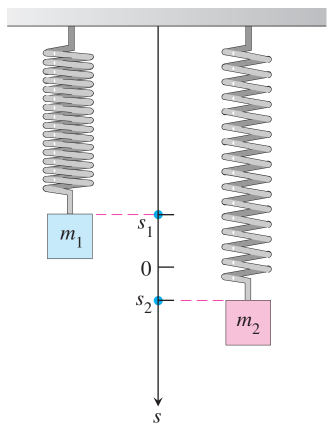
43. Writing to Learn Is the function $f(x) = x^2 - x + 1$ ever negative? Explain. No. It has an absolute minimum at the point $\left(\frac{1}{2}, \frac{3}{4}\right)$.

44. Writing to Learn You have been asked to determine whether the function $f(x) = 3 + 4 \cos x + \cos 2x$ is ever negative.

(a) Explain why you need to consider values of x only in the interval $[0, 2\pi]$. Because $f(x)$ is periodic with period 2π.

(b) Is f ever negative? Explain.
No. It has an absolute minimum at the point $(\pi, 0)$.

45. ***Vertical Motion*** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively, with s_1 and s_2 in meters and t in seconds.



Whenever t is an integer multiple of π sec.

(a) At what times in the interval $t > 0$ do the masses pass each other? [*Hint:* $\sin 2t = 2 \sin t \cos t$.]

The greatest distance is $3\sqrt{3}/2$ m when $t = 2\pi/3$ and $4\pi/3$ sec.

(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)

46. ***Motion on a Line*** The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

At $t = \pi/3$ sec and at $t = 4\pi/3$ sec

(a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

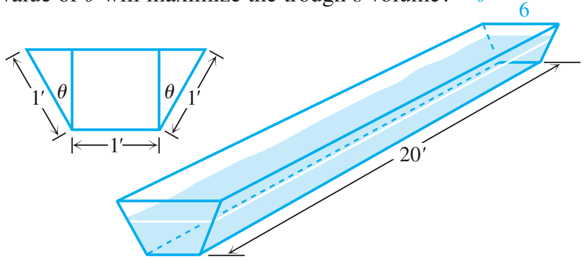
The maximum distance between particles is 1 m.

(b) What is the farthest apart that the particles ever get?

Near $t = \pi/3$ sec and near $t = 4\pi/3$ sec

(c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

47. ***Finding an Angle*** The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume? $\theta = \dfrac{\pi}{6}$

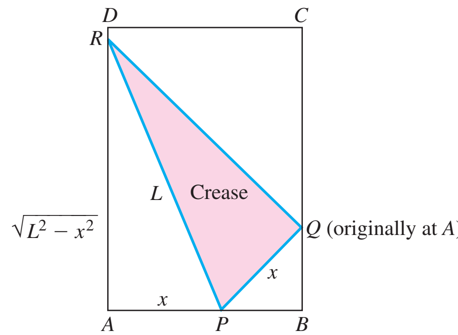


48. **Group Activity** ***Paper Folding*** A rectangular sheet of 8 1/2- by 11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

(a) Show that $L^2 = 2x^3/(2x - 8.5)$. Answers will vary.

(b) What value of x minimizes L^2? $x = \dfrac{51}{8} = 6.375$ in.

(c) What is the minimum value of L? Minimum length ≈ 11.04 in.



49. ***Sensitivity to Medicine*** *(continuation of Exercise 48, Section 3.3)* Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM. $M = \dfrac{C}{2}$

50. ***Selling Backpacks*** It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are certain positive constants. What selling price will bring a maximum profit? $x = \dfrac{c + 100}{2} = 50 + \dfrac{c}{2}$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. True or False A continuous function on a closed interval must attain a maximum value on that interval. Justify your answer. True. This is guaranteed by the Extreme Value Theorem (Section 4.1).

52. True or False If $f'(c) = 0$ and $f(c)$ is not a local maximum, then $f(c)$ is a local minimum. Justify your answer. False. For example, consider $f(x) = x^3$ at $c = 0$.

53. Multiple Choice Two positive numbers have a sum of 60. What is the maximum product of one number times the square of the second number? D

(A) 3481

(B) 3600

(C) 27,000

(D) 32,000

(E) 36,000

54. Multiple Choice A continuous function f has domain [1, 25] and range [3, 30]. If $f'(x) < 0$ for all x between 1 and 25, what is $f(25)$? B

(A) 1

(B) 3

(C) 25

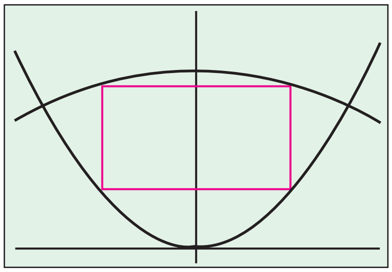
(D) 30

(E) impossible to determine from the information given

55. Multiple Choice What is the maximum area of a right triangle with hypotenuse 10? B

(A) 24 **(B)** 25 **(C)** $25\sqrt{2}$ **(D)** 48 **(E)** 50

56. Multiple Choice A rectangle is inscribed between the parabolas $y = 4x^2$ and $y = 30 - x^2$ as shown below:



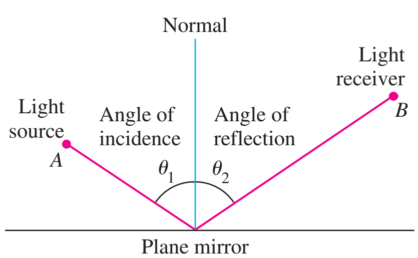
[–3, 3] by [–2, 40]

What is the maximum area of such a rectangle ? E

(A) $20\sqrt{2}$ **(B)** 40 **(C)** $30\sqrt{2}$ **(D)** 50 **(E)** $40\sqrt{2}$

Explorations

57. *Fermat's Principle in Optics* Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



58. *Tin Pest* When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product,

$a =$ the amount of substance at the beginning,

$k =$ a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?
The rate v is maximum when $x = \frac{a}{2}$. The rate then is $\frac{ka^2}{4}$.

59. *How We Cough* When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v (in cm/sec) can be modeled by the equation

$$v = c(r_0 - r)r^2, \quad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in cm and c is a positive constant whose value depends in part on the length of the trachea.

(a) Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

(b) Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \le r \le 0.5$. Compare what you see to the claim that v is a maximum when $r = (2/3)r_0$.

60. *Wilson Lot Size Formula* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

$q = \sqrt{2km/h}$

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now? $q = \sqrt{2km/h}$ (the same amount as in part (a))

61. $p(x) = 6x - (x^3 - 6x^2 + 15x)$, $x \geq 0$. This function has its maximum value at the points (0, 0) and (3, 0).

61. *Production Level* Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

62. *Production Level* Suppose $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items. $x = 10$ items

Extending the Ideas

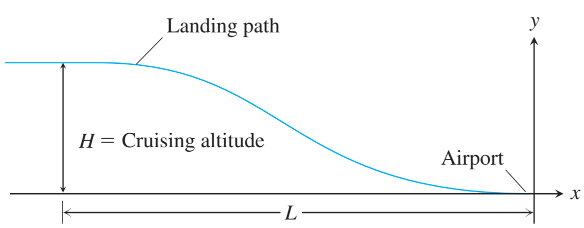
63. *Airplane Landing Path* An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.

(a) What is dy/dx at $x = 0$?

(b) What is dy/dx at $x = -L$?

(c) Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



In Exercises 64 and 65, you might find it helpful to use a CAS.

64. *Generalized Cone Problem* A cone of height h and radius r is constructed from a flat, circular disk of radius a in. as described in Exploration 1.

(a) Find a formula for the volume V of the cone in terms of x and a.

(b) Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.

(c) Writing to Learn Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

4. $A(x) = x(4 - x)$, $0 < x < 4$. $A'(x) = 4 - 2x$, so there is an absolute maximum at $x = 2$. If $x = 2$, then the length of the second side is also 2, so the rectangle with the largest area is a square.

7. Largest volume is $\frac{2450}{27} \approx 90.74$ in^3; dimensions: $\frac{5}{3}$ in. by $\frac{14}{3}$ in. by $\frac{35}{3}$ in.

8. Since $a^2 + b^2 = 400$, Area $= \frac{1}{2}a(400 - a^2)^{1/2}$.

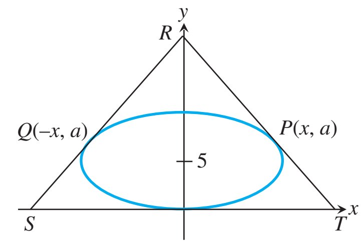
$\frac{d}{da}\text{Area} = \frac{200 - a^2}{(400 - a^2)^{1/2}}$.

Thus the maximum area occurs when $a^2 = 200$, but then $b^2 = 200$ as well, so $a = b$.

65. *Circumscribing an Ellipse* Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at (0, 5). A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



(a) Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

(b) What is the domain of A? Draw the graph of A. How are the asymptotes of the graph related to the problem situation?

(c) Determine the height of the triangle with minimum area. How is it related to the y-coordinate of the center of the ellipse?

(d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

12. (a) $x = 15$ ft and $y = 5$ ft

(b) The material for the tank costs 5 dollars/sq ft and the excavation charge is 10 dollars for each square foot of the cross-sectional area of one wall of the hole.

29. (a) $f'(x)$ is a quadratic polynomial, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(x)$ has no local extrema.

If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(x)$ has 2 local extrema at those points.

(b) Possible answers:

No local extrema: $y = x^3$;

2 local extrema: $y = x^3 - 3x$

34. $f'(x) = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are no local maxima.

4.5 Linearization and Newton's Method

What you'll learn about

- Linear Approximation
- Newton's Method
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change

. . . and why

Engineering and science depend on approximations in most practical applications; it is important to understand how approximation techniques work.

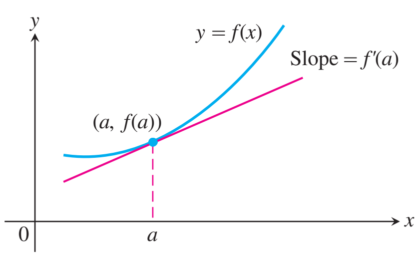


Figure 4.44 The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

Linear Approximation

In our study of the derivative we have frequently referred to the "tangent line to the curve" at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency. We say that differentiable curves are always **locally linear,** a fact that can best be appreciated graphically by zooming in at a point on the curve, as Exploration 1 shows.

EXPLORATION 1 Appreciating Local Linearity

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence "locally linear" there. Let us explore the significance of this fact with the help of a graphing calculator.

1. Graph $y = f(x)$ in the "ZoomDecimal" window. What appears to be the behavior of the function at the point (0, 1)?
2. Show algebraically that f is differentiable at $x = 0$. What is the equation of the tangent line at (0, 1)?
3. Now zoom in repeatedly, keeping the cursor at (0, 1). What is the long-range outcome of repeated zooming?
4. The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

We hope that this exploration gives you a new appreciation for the tangent line. As you zoom in on a differentiable function, its graph at that point actually seems to *become* the graph of the tangent line! This observation—that even the most complicated differentiable curve behaves locally like the simplest graph of all, a straight line—is the basis for most of the applications of differential calculus. It is what allows us, for example, to refer to the derivative as the "slope of the curve" or as "the velocity at time t_0."

Algebraically, the principle of local linearity means that the *equation* of the tangent line defines a function that can be used to *approximate* a differentiable function near the point of tangency. In recognition of this fact, we give the equation of the tangent line a new name: the *linearization of f at a*. Recall that the tangent line at $(a, f(a))$ has point-slope equation $y - f(a) = f'(x)(x - a)$ (Figure 4.44).

DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of f at a.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a.** The point $x = a$ is the **center** of the approximation.

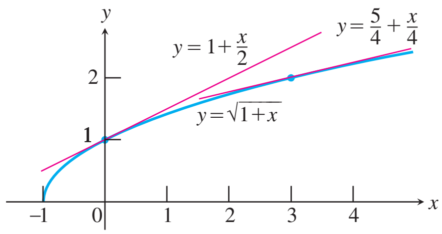


Figure 4.45 The graph of $f(x) = \sqrt{1+x}$ and its linearization at $x = 0$ and $x = 3$. (Example 1)

EXAMPLE 1 Finding a Linearization

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

SOLUTION

Since $f(0) = 1$, the point of tangency is $(0, 1)$. Since $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, the slope of the tangent line is $f'(0) = \frac{1}{2}$. Thus

$$L(x) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}. \quad \text{(Figure 4.45)}$$

To approximate $\sqrt{1.02}$, we use $x = 0.02$:

$$\sqrt{1.02} = f(0.02) \approx L(0.02) = 1 + \frac{0.02}{2} = 1.01$$

The calculator gives $\sqrt{1.02} = 1.009950494$, so the approximation error is $|1.009950494 - 1.01| \approx 4.05 \times 10^{-5}$. We report that the error is less than 10^{-4}.

Now try Exercise 1.

Why not just use a calculator?

We readily admit that linearization will never replace a calculator when it comes to finding square roots. Indeed, historically it was the other way around. Understanding linearization, however, brings you one step closer to understanding how the calculator finds those square roots so easily. You will get many steps closer when you study Taylor polynomials in Chapter 9. (A linearization is just a Taylor polynomial of degree 1.)

Look at how accurate the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ is for values of x near 0.

Approximation	|True Value – Approximation|
$\sqrt{1.002} \approx 1 + \frac{0.002}{2} = 1.001$	$<10^{-6}$
$\sqrt{1.02} \approx 1 + \frac{0.02}{2} = 1.01$	$<10^{-4}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$	$<10^{-2}$

As we move away from zero (the center of the approximation), we lose accuracy and the approximation becomes less useful. For example, using $L(2) = 2$ as an approximation for $f(2) = \sqrt{3}$ is not even accurate to one decimal place. We could do slightly better using $L(2)$ to approximate $f(2)$ if we were to use 3 as the center of our approximation (Figure 4.45).

EXAMPLE 2 Finding a Linearization

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ and use it to approximate cos 1.75 without a calculator. Then use a calculator to determine the accuracy of the approximation.

SOLUTION

Since $f(\pi/2) = \cos(\pi/2) = 0$, the point of tangency is $(\pi/2, 0)$. The slope of the tangent line is $f'(\pi/2) = -\sin(\pi/2) = -1$. Thus

$$L(x) = 0 + (-1)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}. \quad \text{(Figure 4.46)}$$

To approximate cos (1.75), we use $x = 1.75$:

$$\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$$

The calculator gives $\cos 1.75 = -0.1782460556$, so the approximation error is $|-0.1782460556 - (-1.75 + \pi/2)| \approx 9.57 \times 10^{-4}$. We report that the error is less than 10^{-3}.

Now try Exercise 5.

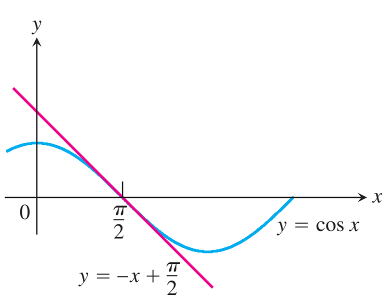


Figure 4.46 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$. (Example 2)

EXAMPLE 3 Approximating Binomial Powers

Example 1 introduces a special case of a general linearization formula that applies to powers of $1 + x$ for small values of x:

$$(1 + x)^k \approx 1 + kx.$$

If k is a positive integer this follows from the Binomial Theorem, but the formula actually holds for *all* real values of k. (We leave the justification to you as Exercise 7.) Use this formula to find polynomials that will approximate the following functions for values of x close to zero:

(a) $\sqrt[3]{1 - x}$ **(b)** $\dfrac{1}{1 - x}$ **(c)** $\sqrt{1 + 5x^4}$ **(d)** $\dfrac{1}{\sqrt{1 - x^2}}$

SOLUTION

We change each expression to the form $(1 + y)^k$, where k is a real number and y is a function of x that is close to 0 when x is close to zero. The approximation is then given by $1 + ky$.

(a) $\sqrt[3]{1 - x} = (1 + (-x))^{1/3} \approx 1 + \dfrac{1}{3}(-x) = 1 - \dfrac{x}{3}$

(b) $\dfrac{1}{1 - x} = (1 + (-x))^{-1} \approx 1 + (-1)(-x) = 1 + x$

(c) $\sqrt{1 + 5x^4} = ((1 + 5x^4))^{1/2} \approx 1 + \dfrac{1}{2}(5x^4) = 1 + \dfrac{5}{2}x^4$

(d) $\dfrac{1}{\sqrt{1 - x^2}} = ((1 + (-x^2))^{-1/2} \approx 1 + \left(-\dfrac{1}{2}\right)(-x^2) = 1 + \dfrac{1}{2}x^2$

Now try Exercise 9.

EXAMPLE 4 Approximating Roots

Use linearizations to approximate **(a)** $\sqrt{123}$ and **(b)** $\sqrt[3]{123}$.

SOLUTION

Part of the analysis is to decide where to center the approximations.

(a) Let $f(x) = \sqrt{x}$. The closest perfect square to 123 is 121, so we center the linearization at $x = 121$. The tangent line at $(121, 11)$ has slope

$$f'(121) = \frac{1}{2}(121)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{121}} = \frac{1}{22}.$$

So

$$\sqrt{121} \approx L(121) = 11 + \frac{1}{22}(123 - 121) = 11.\overline{09}.$$

(b) Let $f(x) = \sqrt[3]{x}$. The closest perfect cube to 123 is 125, so we center the linearization at $x = 125$. The tangent line at $(125, 5)$ has slope

$$f'(125) = \frac{1}{3}(125)^{-2/3} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{125})^2} = \frac{1}{75}.$$

So

$$\sqrt[3]{123} \approx L(123) = 5 + \frac{1}{75}(123 - 125) = 4.97\overline{3}.$$

A calculator shows both approximations to be within 10^{-3} of the actual values.

Now try Exercise 11.

Newton's Method

Newton's method is a numerical technique for approximating a zero of a function with zeros of its linearizations. Under favorable circumstances, the zeros of the linearizations

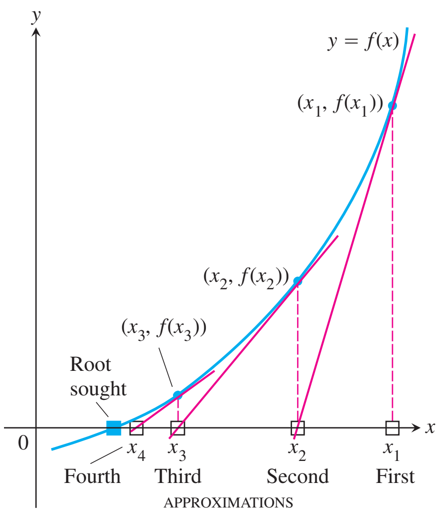


Figure 4.47 Usually the approximations rapidly approach an actual zero of $y = f(x)$.

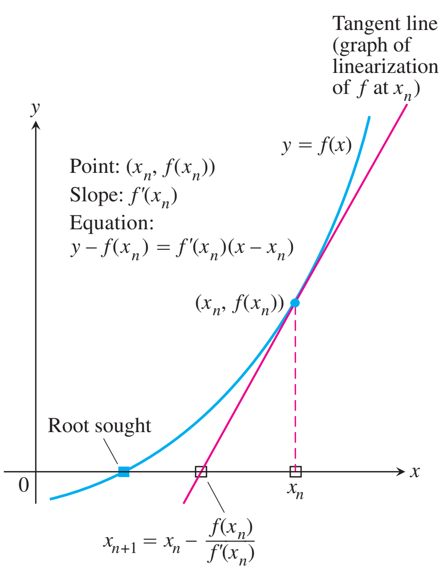


Figure 4.48 From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

converge rapidly to an accurate approximation. Many calculators use the method because it applies to a wide range of functions and usually gets results in only a few steps. Here is how it works.

To find a solution of an equation $f(x) = 0$, we begin with an initial estimate x_1, found either by looking at a graph or simply guessing. Then we use the tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$ to approximate the curve (Figure 4.47). The point where the tangent crosses the x-axis is the next approximation x_2. The number x_2 is usually a better approximation to the solution than is x_1. The point where the tangent to the curve at $(x_2, f(x_2))$ crosses the x-axis is the next approximation x_3. We continue on, using each approximation to generate the next, until we are close enough to the zero to stop.

There is a formula for finding the $(n + 1)$st approximation x_{n+1} from the nth approximation x_n. The point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.48).

$$0 - f(x_n) = f'(x_n)(x - x_n)$$

$$-f(x_n) = f'(x_n) \cdot x - f'(x_n) \cdot x_n$$

$$f'(x_n) \cdot x = f'(x_n) \cdot x_n - f(x_n)$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$

This value of x is the next approximation x_{n+1}. Here is a summary of Newton's method.

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

EXAMPLE 5 Using Newton's Method

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

SOLUTION

Let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}.$$

The graph of f in Figure 4.49 on the next page suggests that $x_1 = -0.3$ is a good first approximation to the zero of f in the interval $-1 \leq x \leq 0$. Then,

$$x_1 = -0.3,$$

$$x_2 = -0.322324159,$$

$$x_3 = -0.3221853603,$$

$$x_4 = -0.3221853546.$$

The x_n for $n \geq 5$ all appear to equal x_4 on the calculator we used for our computations. We conclude that the solution to the equation $x^3 + 3x + 1 = 0$ is about -0.3221853546. *Now try Exercise 15.*

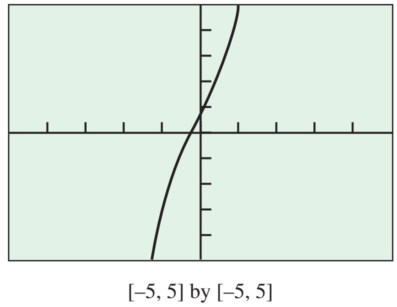


Figure 4.49 A calculator graph of $y = x^3 + 3x + 1$ suggests that -0.3 is a good first guess at the zero to begin Newton's method. (Example 5)

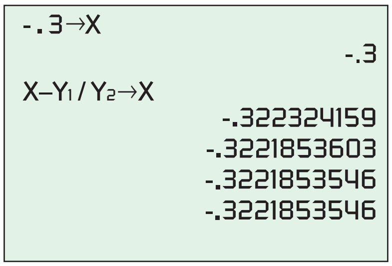


Figure 4.50 A graphing calculator does the computations for Newton's method. (Exploration 2)

EXPLORATION 2 Using Newton's Method on Your Calculator

Here is an easy way to get your calculator to perform the calculations in Newton's method. Try it with the function $f(x) = x^3 + 3x + 1$ from Example 5.

1. Enter the function in Y1 and its derivative in Y2.
2. On the home screen, store the initial guess into x. For example, using the initial guess in Example 5, you would type $-.3 \rightarrow$X.
3. Type X–Y1/Y2→X and press the ENTER key over and over. Watch as the numbers converge to the zero of f. When the values stop changing, it means that your calculator has found the zero to the extent of its displayed digits (Figure 4.50).
4. Experiment with different initial guesses and repeat Steps 2 and 3.
5. Experiment with different functions and repeat Steps 1 through 3. Compare each final value you find with the value given by your calculator's built-in zero-finding feature.

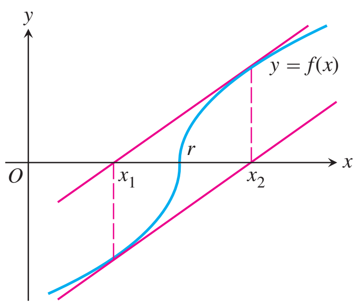


Figure 4.51 The graph of the function

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \ge r. \end{cases}$$

If $x_1 = r - h$, then $x_2 = r + h$. Successive approximations go back and forth between these two values, and Newton's method fails to converge.

Newton's method does not work if $f'(x_1) = 0$. In that case, choose a new starting point.

Newton's method does not always converge. For instance (see Figure 4.51), successive approximations $r - h$ and $r + h$ can go back and forth between these two values, and no amount of iteration will bring us any closer to the zero r.

If Newton's method does converge, it converges to a zero of f. However, the method may converge to a zero that is different from the expected one if the starting value is not close enough to the zero sought. Figure 4.52 shows how this might happen.

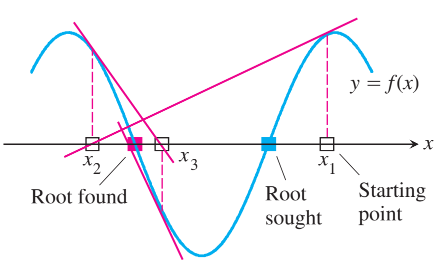


Figure 4.52 Newton's method may miss the zero you want if you start too far away.

Differentials

Leibniz used the notation dy/dx to represent the derivative of y with respect to x. The notation *looks* like a quotient of real numbers, but it is really a *limit* of quotients in which both numerator and denominator go to zero (without actually equaling zero). That makes it tricky to define dy and dx as separate entities. (See the margin note, "Leibniz and His Notation.") Since we really only need to define dy and dx as formal variables, we define them in terms of each other so that their quotient must be the derivative.

Leibniz and His Notation

Although Leibniz did most of his calculus using *dy* and *dx* as separable entities, he never quite settled the issue of what they were. To him, they were "infinitesimals"—nonzero numbers, but infinitesimally small. There was much debate about whether such things could exist in mathematics, but luckily for the early development of calculus it did not matter: thanks to the Chain Rule, *dy*/*dx* *behaved* like a quotient whether it was one or not.

DEFINITION Differentials

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

EXAMPLE 6 Finding the Differential *dy*

Find the differential dy and evaluate dy for the given values of x and dx.

(a) $y = x^5 + 37x, \quad x = 1, \quad dx = 0.01$ **(b)** $y = \sin 3x, \quad x = \pi, \quad dx = -0.02$

(c) $x + y = xy, \quad x = 2, \quad dx = 0.05$

SOLUTION

(a) $dy = (5x^4 + 37)\,dx$. When $x = 1$ and $dx = 0.01$, $dy = (5 + 37)(0.01) = 0.42$.

(b) $dy = (3 \cos 3x)\,dx$. When $x = \pi$ and $dx = -0.02$, $dy = (3 \cos 3\pi)(-0.02) = 0.06$.

(c) We could solve explicitly for y before differentiating, but it is easier to use implicit differentiation:

$$\begin{aligned} d(x + y) &= d(xy) \\ dx + dy &= xdy + ydx && \text{Sum and Product Rules in differential form} \\ dy(1 - x) &= (y - 1)dx \\ dy &= \frac{(y - 1)dx}{1 - x} \end{aligned}$$

When $x = 2$ in the original equation, $2 + y = 2y$, so y is also 2. Therefore

$$dy = \frac{(2 - 1)(0.05)}{(1 - 2)} = -0.05.$$

Now try Exercise 19.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f.** For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u\,du.$$

Fan Chung Graham
(1949–)

"Don't be intimidated!" is Dr. Fan Chung Graham's advice to young women considering careers in mathematics. Fan Chung Graham came to the U.S. from Taiwan to earn a Ph.D. in Mathematics from the University of Pennsylvania. She worked in the field of combinatorics at Bell Labs and Bellcore, and then, in 1994, returned to her alma mater as a Professor of Mathematics. Her research interests include spectral graph theory, discrete geometry, algorithms, and communication networks.

EXAMPLE 7 Finding Differentials of Functions

(a) $d(\tan 2x) = \sec^2 (2x)\, d(2x) = 2 \sec^2 2x\, dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$

Now try Exercise 27.

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and we want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, f and its linearization L at a will change by nearly the same amount (Figure 4.53). Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

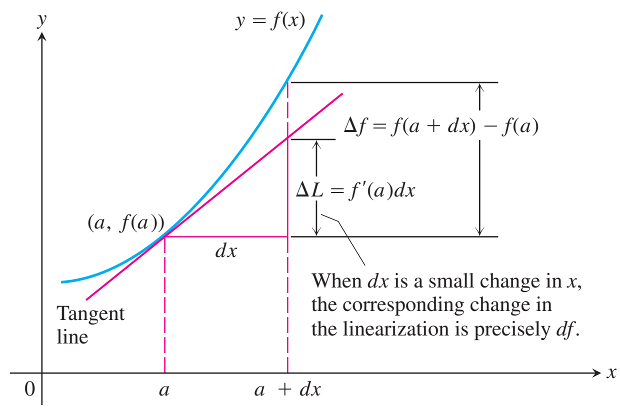


Figure 4.53 Approximating the change in the function f by the change in the linearization of f.

In the notation of Figure 4.53, the change in f is

$$\Delta f = f(a + dx) - f(a).$$

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: The value of df at $x = a$ is ΔL, the change in the linearization of f corresponding to the change dx.

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a)\,dx.$$

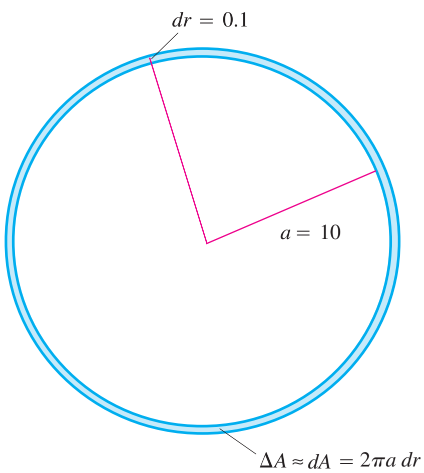


Figure 4.54 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a good estimate of ΔA. (Example 8)

EXAMPLE 8 Estimating Change With Differentials

The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 4.54). Use dA to estimate the increase in the circle's area A. Compare this estimate with the true change ΔA, and find the approximation error.

SOLUTION

Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\, dr = 2\pi a\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = 2.01\pi \text{ m}^2.$$

The approximation error is $\Delta A - dA = 2.01\pi - 2\pi = 0.01\pi \text{ m}^2$.

Now try Exercise 31.

Absolute, Relative, and Percentage Change

As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\, dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

Why It's Easy to Estimate Change in Perimeter

Note that the *true* change in Example 9 is $P(13) - P(12) = 26\pi - 24\pi = 2\pi$, so the differential estimate in this case is perfectly accurate! Why? Since $P = 2\pi r$ is a linear function of r, the linearization of P is the same as P itself. It is useful to keep in mind that local linearity is what makes estimation by differentials work.

EXAMPLE 9 Changing Tires

Inflating a bicycle tire changes its radius from 12 inches to 13 inches. Use differentials to estimate the absolute change, the relative change, and the percentage change in the perimeter of the tire.

SOLUTION

Perimeter $P = 2\pi r$, so $\Delta P \approx dP = 2\pi\, dr = 2\pi(1) = 2\pi \approx 6.28$.

The absolute change is approximately 6.3 inches.

The relative change (when $P(12) = 24\pi$) is approximately $2\pi/24\pi \approx 0.08$.

The percentage change is approximately 8 percent. ***Now try Exercise 35.***

Another way to interpret the change in $f(x)$ resulting from a change in x is the effect that an error in estimating x has on the estimation of $f(x)$. We illustrate this in Example 10.

EXAMPLE 10 Estimating the Earth's Surface Area

Suppose the earth were a perfect sphere and we determined its radius to be 3959 ± 0.1 miles. What effect would the tolerance of ± 0.1 mi have on our estimate of the earth's surface area?

continued

SOLUTION

The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is

$$dS = 8\pi r\, dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland.

Now try Exercise 41.

EXAMPLE 11 Determining Tolerance

About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

SOLUTION

We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \le \frac{1}{100}S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality by its approximation

$$dS = \left(\frac{dS}{dr}\right)dr = 8\pi r\, dr.$$

This gives

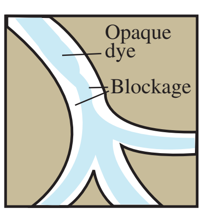
$$|8\pi r\, dr| \le \frac{4\pi r^2}{100}, \quad \text{or} \quad |dr| \le \frac{1}{8\pi r}\cdot\frac{4\pi r^2}{100} = \frac{1}{2}\cdot\frac{r}{100} = 0.005r.$$

We should measure r with an error dr that is no more than 0.5% of the true value.

Now try Exercise 49.

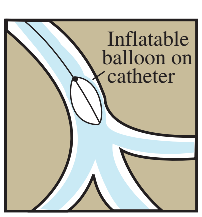
Angiography

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.



Angioplasty

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.



EXAMPLE 12 Unclogging Arteries

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

SOLUTION

The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}dr = 4kr^3\, dr.$$

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow.

Now try Exercise 51.

Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx.

EXAMPLE 13 Finding Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

SOLUTION

The size of ds in the equation

$$ds = 32t\,dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

Now try Exercise 53.

Quick Review 4.5 *(For help, go to Sections 3.3, 3.6, and 3.9.)*

In Exercises 1 and 2, find dy/dx.

1. $y = \sin(x^2 + 1)$ $2x\cos(x^2+1)$

2. $y = \dfrac{x + \cos x}{x + 1}$ $\dfrac{1 - \cos x - (x+1)\sin x}{(x+1)^2}$

In Exercises 3 and 4, solve the equation graphically.

3. $xe^{-x} + 1 = 0$ $x \approx -0.567$

4. $x^3 + 3x + 1 = 0$ $x \approx -0.322$

In Exercises 5 and 6, let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to f at $x = c$.

5. $c = 0$ $y = x + 1$

6. $c = -1$ $y = 2ex + e + 1$

7. Find where the tangent line in **(a)** Exercise 5 and **(b)** Exercise 6 crosses the x-axis. **(a)** $x = -1$ **(b)** $x = -\dfrac{e+1}{2e} \approx -0.684$

8. Let $g(x)$ be the function whose graph is the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at $x = 1$. Complete the table.

x	$f(x)$	$g(x)$
0.7	−1.457	−1.7
0.8	−1.688	−1.8
0.9	−1.871	−1.9
1	−2	−2
1.1	−2.069	−2.1
1.2	−2.072	−2.2
1.3	−2.003	−2.3

In Exercises 9 and 10, graph $y = f(x)$ and its tangent line at $x = c$.

9. $c = 1.5$, $f(x) = \sin x$

10. $c = 4$, $f(x) = \begin{cases} -\sqrt{3 - x}, & x < 3 \\ \sqrt{x - 3}, & x \ge 3 \end{cases}$

Section 4.5 Exercises

In Exercises 1–6, **(a)** find the linearization $L(x)$ of $f(x)$ at $x = a$. **(b)** How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

1. $f(x) = x^3 - 2x + 3,\quad a = 2$

2. $f(x) = \sqrt{x^2 + 9},\quad a = -4$

3. $f(x) = x + \dfrac{1}{x},\quad a = 1$

4. $f(x) = \ln(x + 1),\quad a = 0$

5. $f(x) = \tan x,\quad a = \pi$

6. $f(x) = \cos^{-1} x,\quad a = 0$

7. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

8. Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.

(a) $(1.002)^{100}$ $\approx 1.2,\ |1.002^{100} - 1.2| < 10^{-1}$

(b) $\sqrt[3]{1.009}$ $\approx 1.003,\ |\sqrt[3]{1.009} - 1.003| < 10^{-5}$

In Exercises 9 and 10, use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

9. **(a)** $f(x) = (1 - x)^6$ $1 - 6x$ **(b)** $f(x) = \dfrac{2}{1 - x}$ $2 + 2x$ **(c)** $f(x) = \dfrac{1}{\sqrt{1 + x}}$ $1 - \dfrac{x}{2}$

11. $y = 10 + 0.05(x - 10)$, so $y = 10.05$
12. $y = 3 + (1/27)(x - 27)$, so $y = 3 - 1/27 = 2.9\overline{62}$
13. $y = 10 + (1/300)(x - 1000)$, so $y = 10 - 1/150 = 9.99\overline{3}$
14. $y = 9 + (1/18)(x - 81)$, so $y = 9 - 1/18 = 8.9\overline{4}$

10. **(a)** $f(x) = (4 + 3x)^{1/3}$ **(b)** $f(x) = \sqrt{2 + x^2}$

(c) $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

(a) $\sqrt{2}\left(1 + \dfrac{x^2}{4}\right)$ **(b)** $4^{1/3}\left(1 + \dfrac{x}{4}\right)$ **(c)** $1 - \dfrac{2}{6 + 3x}$

In Exercises 11–14, approximate the root by using a linearization centered at an appropriate nearby number.

11. $\sqrt{101}$ **12.** $\sqrt[3]{26}$

13. $\sqrt[3]{998}$ **14.** $\sqrt{80}$

In Exercises 15–18, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

15. $x^3 + x - 1 = 0$ $x \approx 0.682328$ **16.** $x^4 + x - 3 = 0$ $x \approx -1.452627, 1.164035$

17. $x^2 - 2x + 1 = \sin x$ $x \approx 0.386237, 1.961569$ **18.** $x^4 - 2 = 0$ $x \approx \pm 1.189207$

In Exercises 19–26, **(a)** find dy, and **(b)** evaluate dy for the given value of x and dx.

19. $y = x^3 - 3x, \quad x = 2, \quad dx = 0.05$

20. $y = \dfrac{2x}{1 + x^2}, \quad x = -2, \quad dx = 0.1$

21. $y = x^2 \ln x, \quad x = 1, \quad dx = 0.01$

22. $y = x\sqrt{1 - x^2}, \quad x = 0, \quad dx = -0.2$

23. $y = e^{\sin x}, \quad x = \pi, \quad dx = -0.1$

24. $y = 3 \csc\left(1 - \dfrac{x}{3}\right), \quad x = 1, \quad dx = 0.1$

25. $y + xy - x = 0, \quad x = 0, \quad dx = 0.01$

26. $2y = x^2 - xy, \quad x = 2, \quad dx = -0.05$

In Exercises 27–30, find the differential.

27. $d(\sqrt{1-x^2})$ $-\dfrac{x}{\sqrt{1 - x^2}}dx$

28. $d(e^{5x} + x^5)$ $(5e^{5x} + 5x^4)\,dx$

29. $d(\arctan 4x)$ $\dfrac{4}{1 + 16x^2}dx$

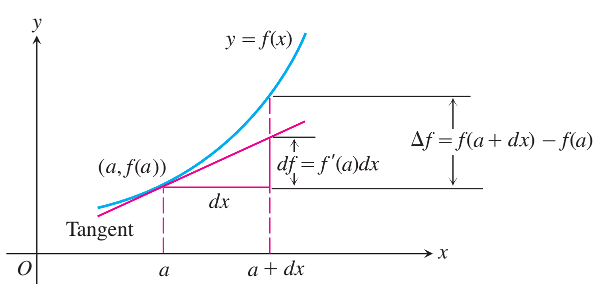
30. $d(8^x + x^8)$ $(8^x \ln 8 + 8x^7)dx$

In Exercises 31–34, the function f changes value when x changes from a to $a + dx$. Find

(a) the true change $\Delta f = f(a + dx) - f(a)$.

(b) the estimated change $df = f'(a)\,dx$.

(c) the approximation error $|\Delta f - df|$.



31. $f(x) = x^2 + 2x, \quad a = 0, \quad dx = 0.1$ **(a)** 0.21 **(b)** 0.2 **(c)** 0.01

32. $f(x) = x^3 - x, \quad a = 1, \quad dx = 0.1$ **(a)** 0.231 **(b)** 0.2 **(c)** 0.031

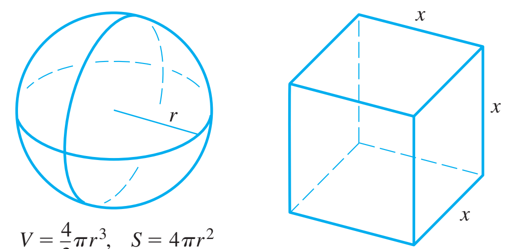
33. $f(x) = x^{-1}, \quad a = 0.5, \quad dx = 0.05$ **(a)** $-\dfrac{2}{11}$ **(b)** $-\dfrac{1}{5}$ **(c)** $\dfrac{1}{55}$

34. $f(x) = x^4, \quad a = 1, \quad dx = 0.01$ **(a)** 0.04060401 **(b)** 0.04 **(c)** 0.00060401

In Exercises 35–40, write a differential formula that estimates the given change in volume or surface area. Then use the formula to estimate the change when the dependent variable changes from 10 cm to 10.05 cm.

35. ***Volume*** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from a to $a + dr$ $\Delta V \approx 4\pi a^2\,dr = 20\pi\ \text{cm}^3$

36. ***Surface Area*** The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from a to $a + dr$ $\Delta A \approx 8\pi a\,dr = 4\pi\ \text{cm}^2$



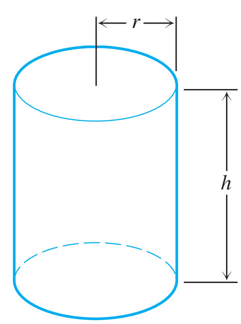
$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$ $V = x^3, \quad S = 6x^2$

37. ***Volume*** The change in the volume $V = x^3$ of a cube when the edge lengths change from a to $a + dx$ $\Delta V \approx 3a^2\,dx = 15\ \text{cm}^3$

38. ***Surface Area*** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from a to $a + dx$ $\Delta S \approx 12a\,dx = 6\ \text{cm}^2$

39. ***Volume*** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from a to $a + dr$ and the height does not change $\Delta V \approx 2\pi ah\,dr = \pi h\ \text{cm}^3$

40. ***Surface Area*** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from a to $a + dh$ and the radius does not change $\Delta A \approx 2\pi r\,dh = 0.1\pi r\ \text{cm}^2$



$V = \pi r^2 h, \quad S = 2\pi rh$

In Exercises 41–44, use differentials to estimate the maximum error in measurement resulting from the tolerance of error in the dependent variable. Express answers to the nearest tenth, since that is the precision used to express the tolerance.

41. The area of a circle with radius 10 ± 0.1 in. $2\pi(10)(0.1) \approx 6.3\ \text{in}^2$

42. The volume of a sphere with radius 8 ± 0.3 in. $4\pi(8)^2(0.3) \approx 241\ \text{in}^2$

43. The volume of a cube with side 15 ± 0.2 cm. $3(15)^2(0.2) \approx 135\ \text{cm}^2$

$(\sqrt{3}/2)(20)(0.5) \approx 8.7 \text{ cm}^2$

44. The area of an equilateral triangle with side 20 ± 0.5 cm.

45. ***Linear Approximation*** Let f be a function with $f(0) = 1$ and $f'(x) = \cos(x^2)$.

(a) Find the linearization of f at $x = 0$. $x + 1$

(b) Estimate the value of f at $x = 0.1$. $f(0.1) \approx 1.1$

(c) **Writing to Learn** Do you think the actual value of f at $x = 0.1$ is greater than or less than the estimate in part (b)? Explain. The actual value is less than 1.1, since the derivative is decreasing over the interval [0, 0.1].

46. ***Expanding Circle*** The radius of a circle is increased from 2.00 to 2.02 m.

(a) Estimate the resulting change in area. $0.08\pi \approx 0.2513$

(b) Estimate as a percentage of the circle's original area. 2%

47. ***Growing Tree*** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross section area?

48. ***Percentage Error*** The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation. 3%

49. ***Tolerance*** About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value? The side should be measured to within 1%.

50. ***Tolerance*** **(a)** About how accurately must the interior diameter of a 10-m high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value? Within 0.5%

(b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount? Within 5%

51. ***Minting Coins*** A manufacturer contracts to mint coins for the federal government. The coins must weigh within 0.1% of their ideal weight, so the volume must be within 0.1% of the ideal volume. Assuming the thickness of the coins does not change, what is the percentage change in the volume of the coin that would result from a 0.1% increase in the radius?

52. ***Tolerance*** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

53. ***Estimating Volume*** You can estimate the volume of a sphere by measuring its circumference with a tape measure, dividing by 2π to get the radius, then using the radius in the volume formula. Find how sensitive your volume estimate is to a 1/8 in. error in the circumference measurement by filling in the table below for spheres of the given sizes. Use differentials when filling in the last column.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	1/8 in.		
Melon	4 in.	1/8 in.		
Beach Ball	7 in.	1/8 in.		

47. The diameter grew $\frac{2}{\pi} \approx 0.6366$ in. The cross section area grew about 10 in^2.

51. $V = \pi r^2 h$ (where h is constant), so $\frac{dV}{V} = \frac{2\pi rh dr}{\pi r^2 h} = 2\frac{dr}{r} = 0.2\%$

54. ***Estimating Surface Area*** Change the heading in the last column of the table in Exercise 53 to "Surface Area Error" and find how sensitive the measure of surface area is to a 1/8 in. error in estimating the circumference of the sphere.

55. ***The Effect of Flight Maneuvers on the Heart*** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \ (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2 \text{ ft/sec}^2$, with the effect the same change dg would have on Earth, where $g = 32 \text{ ft/sec}^2$. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}. About 37.87 to 1

56. ***Measuring Acceleration of Gravity*** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

$dT = -\pi L^{1/2} g^{-3/2}\, dg$

(a) With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).

(b) **Writing to Learn** If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain. If g increases, T decreases and the clock speeds up. This can be seen from the fact that dT and dg have opposite signs.

(c) A clock with a 100-cm pendulum is moved from a location where $g = 980 \text{ cm/sec}^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location. $dg \approx -0.9765$, so $g \approx 979.0235$

52. The height should be measured to within $\frac{1}{3}\%$

57. True. A look at the graph reveals the problem. The graph decreases after $x = 1$ toward a horizontal asymptote of $x = 0$, so the x-intercepts of the tangent lines keep getting bigger without approaching a zero.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

57. True or False Newton's method will not find the zero of $f(x) = x/(x^2 + 1)$ if the first guess is greater than 1. Justify your answer.

58. True or False If u and v are differentiable functions, then $d(uv) = du\, dv$. Justify your answer. False. By the product rule, $d(uv) = u\,dv + v\,du$.

59. Multiple Choice What is the linearization of $f(x) = e^x$ at $x = 1$? B

(A) $y = e$ **(B)** $y = ex$ **(C)** $y = e^x$
(D) $y = x - e$ **(E)** $y = e(x - 1)$

60. Multiple Choice If $y = \tan x$, $x = \pi$, and $dx = 0.5$, what does dy equal? A

(A) -0.25 **(B)** -0.5 **(C)** 0 **(D)** 0.5 **(E)** 0.25

61. Multiple Choice If Newton's method is used to find the zero of $f(x) = x - x^3 + 2$, what is the third estimate if the first estimate is 1? D

(A) $-\frac{3}{4}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{8}{5}$ **(D)** $\frac{18}{11}$ **(E)** 3

62. Multiple Choice If the linearization of $y = \sqrt[3]{x}$ at $x = 64$ is used to approximate $\sqrt[3]{66}$, what is the percentage error? A

(A) 0.01% **(B)** 0.04% **(C)** 0.4% **(D)** 1% **(E)** 4%

Explorations

63. *Newton's Method* Suppose your first guess in using Newton's method is lucky in the sense that x_1 is a root of $f(x) = 0$. What happens to x_2 and later approximations?

64. *Oscillation* Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

65. *Approximations that Get Worse and Worse* Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate x_2, x_3, x_4, and x_5. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

66. *Quadratic Approximations*

(a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$,

ii. $Q'(a) = f'(a)$,

iii. $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

(b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

(c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

(d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

(e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

(f) What are the linearizations of f, g, and h at the respective points in parts (b), (d), and (e)?

67. *Multiples of Pi* Store any number as X in your calculator. Then enter the command X−tan(X)→X and press the ENTER key repeatedly until the displayed value stops changing. The result is always an integral multiple of π. Why is this so? [*Hint:* These are zeros of the sine function.]

Extending the Ideas

68. *Formulas for Differentials* Verify the following formulas. Just multiply the corresponding derivative formulas by dx.

(a) $d(c) = 0$ (c a constant)

(b) $d(cu) = c\,du$ (c a constant)

(c) $d(u + v) = du + dv$

(d) $d(u \cdot v) = u\,dv + v\,du$

(e) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

(f) $d(u^n) = nu^{n-1}\,du$

69. *Linearization* Show that the approximation of $\tan x$ by its linearization at the origin must improve as $x \to 0$ by showing that

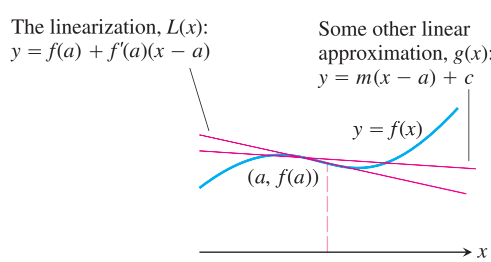
$$\lim_{x\to 0} \frac{\tan x}{x} = 1.$$

70. *The Linearization is the Best Linear Approximation* Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ (m and c constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

i. $E(a) = 0$, The error is zero at $x = a$.

ii. $\lim_{x\to a} \frac{E(x)}{x - a} = 0$, The error is negligible when compared with $(x - a)$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.



71. Writing to Learn Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x + 1}$ and $\sin x$?

63. If $f'(x_1) \neq 0$, then x_2 and all later approximations are equal to x_1.

The linearization is $1 + \frac{3x}{2}$. It is the sum of the two individual linearizations.

4.6 Related Rates

What you'll learn about

- Related Rate Equations
- Solution Strategy
- Simulating Related Motion

. . . and why

Related rate problems are at the heart of Newtonian mechanics; it was essentially to solve such problems that calculus was invented.

Related Rate Equations

Suppose that a particle $P(x, y)$ is moving along a curve C in the plane so that its coordinates x and y are differentiable functions of time t. If D is the distance from the origin to P, then using the Chain Rule we can find an equation that relates dD/dt, dx/dt, and dy/dt.

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

Any equation involving two or more variables that are differentiable functions of time t can be used to find an equation that relates their corresponding rates.

EXAMPLE 1 Finding Related Rate Equations

(a) Assume that the radius r of a sphere is a differentiable function of t and let V be the volume of the sphere. Find an equation that relates dV/dt and dr/dt.

(b) Assume that the radius r and height h of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

SOLUTION

(a) $V = \frac{4}{3}\pi r^3$ Volume formula for a sphere

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(b) $V = \frac{\pi}{3}r^2 h$ Cone volume formula

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2r\frac{dr}{dt} \cdot h\right) = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Now try Exercise 3.

Solution Strategy

What has always distinguished calculus from algebra is its ability to deal with variables that change over time. Example 1 illustrates how easy it is to move from a formula relating static variables to a formula that relates their rates of change: simply differentiate the formula implicitly with respect to t. This introduces an important category of problems called *related rate problems* that still constitutes one of the most important applications of calculus.

We introduce a strategy for solving related rate problems, similar to the strategy we introduced for max-min problems earlier in this chapter.

Strategy for Solving Related Rate Problems

1. **Understand the problem.** In particular, identify the variable whose rate of change you *seek* and the variable (or variables) whose rate of change you *know*.

2. **Develop a mathematical model of the problem.** Draw a picture (many of these problems involve geometric figures) and label the parts that are important to the problem. *Be sure to distinguish constant quantities from variables that change over time.* Only constant quantities can be assigned numerical values at the start.

continued

3. **Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.** The formula is often geometric, but it could come from a scientific application.
4. **Differentiate both sides of the equation implicitly with respect to time *t*.** Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.
5. **Substitute values for any quantities that depend on time.** Notice that it is only safe to do this *after* the differentiation step. Substituting too soon "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.
6. **Interpret the solution.** Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

We illustrate the strategy in Example 2.

EXAMPLE 2 A Rising Balloon

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

SOLUTION

We will carefully identify the six steps of the strategy in this first example.

Step 1: Let h be the height of the balloon and let θ be the elevation angle.

We seek: dh/dt

We know: $d\theta/dt = 0.14$ rad/min

Step 2: We draw a picture (Figure 4.55). We label the horizontal distance "500 ft" because it does not change over time. We label the height "h" and the angle of elevation "θ." Notice that we do not label the angle "$\pi/4$," as that would freeze the picture.

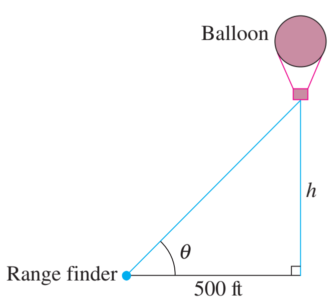


Figure 4.55 The picture shows how h and θ are related geometrically. We seek dh/dt when $\theta = \pi/4$ and $d\theta/dt = 0.14$ rad/min. (Example 2)

Step 3: We need a formula that relates h and θ. Since $\dfrac{h}{500} = \tan\theta$, we get $h = 500\tan\theta$.

Step 4: Differentiate implicitly:

$$\frac{d}{dt}(h) = \frac{d}{dt}(500\tan\theta)$$
$$\frac{dh}{dt} = 500\sec^2\theta\,\frac{d\theta}{dt}$$

Step 5: Let $d\theta/dt = 0.14$ and let $\theta = \pi/4$. (Note that it is now safe to specify our moment in time.)

$$\frac{dh}{dt} = 500\sec^2\left(\frac{\pi}{4}\right)(0.14) = 500(\sqrt{2})^2(0.14) = 140.$$

Step 6: At the moment in question, the balloon is rising at the rate of 140 ft/min.

Now try Exercise 11.

Unit Analysis in Example 2

A careful analysis of the units in Example 2 gives

$$dh/dt = (500 \text{ ft})(\sqrt{2})^2(0.14 \text{ rad/min}) = 140 \text{ ft}\cdot\text{rad/min}.$$

Remember that radian measure is actually dimensionless, adaptable to whatever unit is applied to the "unit" circle. The linear units in Example 2 are measured in feet, so "ft · rad" is simply "ft."

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

continued

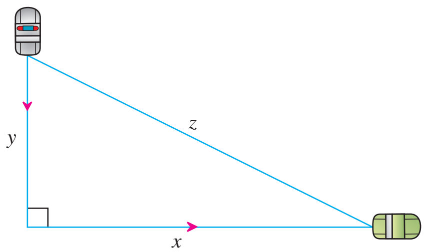


Figure 4.56 A sketch showing the variables in Example 3. We know dy/dt and dz/dt, and we seek dx/dt. The variables x, y, and z are related by the Pythagorean Theorem: $x^2 + y^2 = z^2$.

SOLUTION

We carry out the steps of the strategy.

Let x be the distance of the speeding car from the intersection, let y be the distance of the police cruiser from the intersection, and let z be the distance between the car and the cruiser. Distances x and z are increasing, but distance y is decreasing; so dy/dt is negative.

We seek: dx/dt

We know: $dz/dt = 20$ mph and $dy/dt = -60$ mph

A sketch (Figure 4.56) shows that x, y, and z form three sides of a right triangle. We need to relate those three variables, so we use the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating implicitly with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}, \text{ which reduces to } x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

We now substitute the numerical values for x, y, dz/dt, dy/dt, and z (which equals $\sqrt{x^2 + y^2}$):

$$(0.8)\frac{dx}{dt} + (0.6)(-60) = \sqrt{(0.8)^2 + (0.6)^2}(20)$$

$$(0.8)\frac{dx}{dt} - 36 = (1)(20)$$

$$\frac{dx}{dt} = 70$$

At the moment in question, the car's speed is 70 mph. ***Now try Exercise 13.***

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

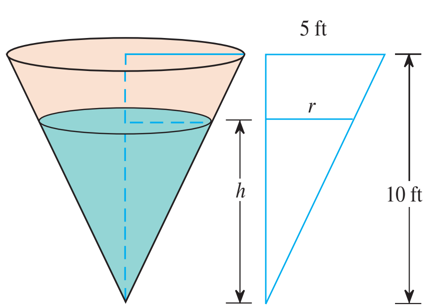


Figure 4.57 In Example 4, the cone of water is increasing in volume inside the reservoir. We know dV/dt and we seek dh/dt. Similar triangles enable us to relate V directly to h.

SOLUTION 1

We carry out the steps of the strategy. Figure 4.57 shows a partially filled conical tank. The tank itself does not change over time; what we are interested in is the changing cone of *water* inside the tank. Let V be the volume, r the radius, and h the height of the cone of water.

We seek: dh/dt

We know: $dV/dt = 9$ ft^3/min

We need to relate V and h. The volume of the cone of water is $V = \frac{1}{3}\pi r^2 h$, but this formula also involves the variable r, whose rate of change is not given. We need to either find dr/dt (see Solution 2) or eliminate r from the equation, which we can do by using the similar triangles in Figure 4.57 to relate r and h:

$$\frac{r}{h} = \frac{5}{10}, \text{ or simply } r = \frac{h}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

continued

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Let $h = 6$ and $dV/dt = 9$; then solve for dh/dt:

$$9 = \frac{\pi}{4}(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at 0.32 ft/min.

SOLUTION 2

The similar triangle relationship

$$r = \frac{h}{2} \text{ also implies that } \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

and that $r = 3$ when $h = 6$. So, we could have left all three variables in the formula $V = \frac{1}{3}\pi r^2 h$ and proceeded as follows:

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

$$= \frac{1}{3}\pi\left(2r\left(\frac{1}{2}\frac{dh}{dt}\right)h + r^2\frac{dh}{dt}\right)$$

$$9 = \frac{1}{3}\pi\left(2(3)\left(\frac{1}{2}\frac{dh}{dt}\right)(6) + (3)^2\frac{dh}{dt}\right)$$

$$9 = 9\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi}$$

This is obviously more complicated than the one-variable approach. In general, it is computationally easier to simplify expressions as much as possible *before* you differentiate.

Now try Exercise 17.

Simulating Related Motion

Parametric mode on a grapher can be used to simulate the motion of moving objects when the motion of each can be expressed as a function of time. In a classic related rate problem, the top end of a ladder slides vertically down a wall as the bottom end is pulled horizontally away from the wall at a steady rate. Exploration 1 shows how you can use your grapher to simulate the related movements of the two ends of the ladder.

EXPLORATION 1 **The Sliding Ladder**

A 10-foot ladder leans against a vertical wall. The base of the ladder is pulled away from the wall at a constant rate of 2 ft/sec.

1. Explain why the motion of the two ends of the ladder can be represented by the parametric equations given on the next page.

continued

$$X1T = 2T$$
$$Y1T = 0$$
$$X2T = 0$$
$$Y2T = \sqrt{10^2 - (2T)^2}$$

2. What minimum and maximum values of T make sense in this problem?

3. Put your grapher in parametric and simultaneous modes. Enter the parametric equations and change the graphing style to "0" (the little ball) if your grapher has this feature. Set Tmin=0, Tmax=5, Tstep=5/20, Xmin=−1, Xmax=17, Xscl=0, Ymin=−1, Ymax=11, and Yscl=0. You can speed up the action by making the denominator in the Tstep smaller or slow it down by making it larger.

4. Press GRAPH and watch the two ends of the ladder move as time changes. Do both ends seem to move at a constant rate?

5. To see the simulation again, enter "ClrDraw" from the DRAW menu.

6. If y represents the vertical height of the top of the ladder and x the distance of the bottom from the wall, relate y and x and find dy/dt in terms of x and y. (Remember that $dx/dt = 2$.)

7. Find dy/dt when $t = 3$ and interpret its meaning. Why is it negative?

8. In theory, how fast is the top of the ladder moving as it hits the ground?

Figure 4.58 shows you how to write a calculator program that animates the falling ladder as a line segment.

```
PROGRAM:LADDER
:For (A, 0, 5, .25)
:ClrDraw
:Line(2,2+√ (100–
(2A)²), 2+2A, 2)
:If A=0:Pause
:End
```

```
WINDOW
  Xmin=2
  Xmax=20
  Xscl=0
  Ymin=1
  Ymax=13
  Yscl=0
  Xres=1
```

Figure 4.58 This 5-step program (with the viewing window set as shown) will animate the ladder in Exploration 1. Be sure any functions in the "Y=" register are turned off. Run the program and the ladder appears against the wall; push ENTER to start the bottom moving away from the wall.
For an enhanced picture, you can insert the commands ":Pt-On(2,2+$\sqrt{(100-(2A)^2)}$,2)" and ":Pt-On(2+2A,2,2)" on either side of the middle line of the program.

Quick Review 4.6 *(For help, go to Sections 1.1, 1.4, and 3.7.)*

In Exercises 1 and 2, find the distance between the points A and B.

1. $A(0, 5)$, $B(7, 0)$ $\sqrt{74}$

2. $A(0, a)$, $B(b, 0)$ $\sqrt{a^2 + b^2}$

In Exercises 3–6, find dy/dx.

3. $2xy + y^2 = x + y$ $\dfrac{1 - 2y}{2x + 2y - 1}$

4. $x \sin y = 1 - xy$ $-\dfrac{y + \sin y}{x + x\cos y}$

5. $x^2 = \tan y$ $2x\cos^2 y$

6. $\ln (x + y) = 2x$ $2x + 2y - 1$

In Exercises 7 and 8, find a parametrization for the line segment with endpoints A and B.

7. $A(-2, 1)$, $B(4, -3)$

8. $A(0, -4)$, $B(5, 0)$

7. One possible answer: $x = -2 + 6t$, $y = 1 - 4t$, $0 \le t \le 1$.

8. One possible answer: $x = 5t$, $y = -4 + 4t$, $0 \le t \le 1$.

In Exercises 9 and 10, let $x = 2\cos t$, $y = 2\sin t$. Find a parameter interval that produces the indicated portion of the graph.

9. The portion in the second and third quadrants, including the points on the axes. One possible answer: $\pi/2 \le t \le 3\pi/2$

10. The portion in the fourth quadrant, including the points on the axes. One possible answer: $3\pi/2 \le t \le 2\pi$

4. **(a)** $\frac{dP}{dt} = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$ **(b)** $0 = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$, or, $\frac{dR}{dt} = \left(-\frac{2R}{I}\right)\left(\frac{dI}{dt}\right) = \left(-\frac{2P}{I^3}\right)\left(\frac{dI}{dt}\right)$

Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of t.

1. *Area* The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt. $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

2. *Surface Area* The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt. $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

3. *Volume* The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

(a) How is dV/dt related to dh/dt if r is constant? $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$

(b) How is dV/dt related to dr/dt if h is constant? $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$

(c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant? $\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$

4. *Electrical Power* The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

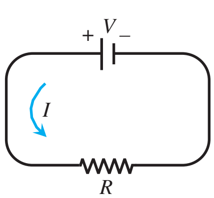
(a) How is dP/dt related to dR/dt and dI/dt?

(b) How is dR/dt related to dI/dt if P is constant?

5. *Diagonals* If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt? See below.

6. *Area* If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\,ab\sin\theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$? $\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$

7. *Changing Voltage* The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in sec.



(a) 1 volt/sec
(b) $-\frac{1}{3}$ amp/sec
(c) $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$
(d) $\frac{dR}{dt} = \frac{3}{2}$ ohms/sec. R is increasing since $\frac{dR}{dt}$ is positive.

(a) What is the value of dV/dt?

(b) What is the value of dI/dt?

(c) Write an equation that relates dR/dt to dV/dt and dI/dt.

(d) Writing to Learn Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

8. *Heating a Plate* When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm? π cm^2/sec

5. $\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$

9. *Changing Dimensions in a Rectangle* The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of See page 255.

(a) the area, **(b)** the perimeter, and

(c) the length of a diagonal of the rectangle.

(d) Writing to Learn Which of these quantities are decreasing, and which are increasing? Explain.

10. *Changing Dimensions in a Rectangular Box* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates: **(a)** 2 m^3/sec **(b)** 0 m^2/sec **(c)** 0 m/sec

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

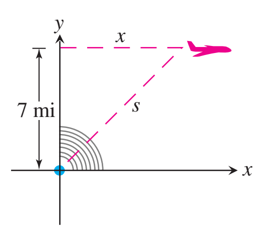
11. *Inflating Balloon* A spherical balloon is inflated with helium at the rate of 100π ft^3/min.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft? 1 ft/min

(b) How fast is the surface area increasing at that instant? 40π ft^2/min

12. *Growing Raindrop* Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate. See page 255.

13. *Air Traffic Control* An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment? $\frac{dx}{dt} = \frac{3000}{\sqrt{51}}$ mph ≈ 420.08 mph



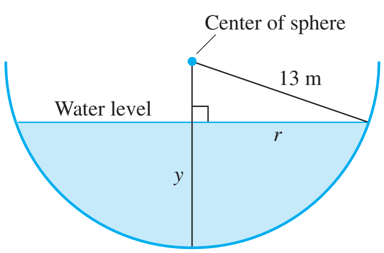
14. *Flying a Kite* Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her? 20 ft/sec

15. *Boring a Cylinder* The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.? $\frac{19\pi}{2500} \approx 0.0239$ in^3/min

16. *Growing Sand Pile* Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.
(a) $\frac{1125}{32\pi} \approx 11.19$ cm/min **(b)** $\frac{375}{8\pi} \approx 14.92$ cm/min

17. *Draining Conical Reservoir* Water is flowing at the rate of 50 m^3/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.
(a) $\frac{32}{9\pi} \approx 1.13$ cm/min **(b)** $-\frac{80}{3\pi} \approx -8.49$ cm/min

18. *Draining Hemispherical Reservoir* Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.

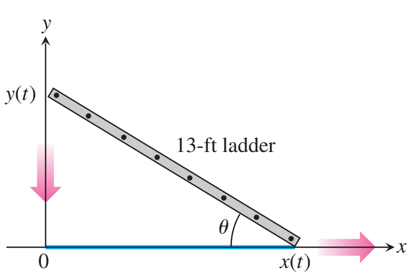


(a) At what rate is the water level changing when the water is 8 m deep? $-\frac{1}{24\pi} \approx -0.01326$ m/min or $-\frac{25}{6\pi} \approx -1.326$ cm/min

(b) What is the radius r of the water's surface when the water is y m deep? $r = \sqrt{169 - (13 - y)^2} = \sqrt{26y - y^2}$

(c) At what rate is the radius r changing when the water is 8 m deep?
$-\frac{5}{288\pi} \approx -0.00553$ m/min or $-\frac{125}{72\pi} \approx -0.553$ cm/min

19. *Sliding Ladder* A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

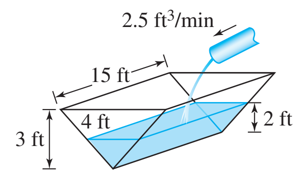


(a) How fast is the top of the ladder sliding down the wall at that moment? 12 ft/sec

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment? $-\frac{119}{2}$ ft^2/sec

(c) At what rate is the angle θ between the ladder and the ground changing at that moment? -1 radian/sec

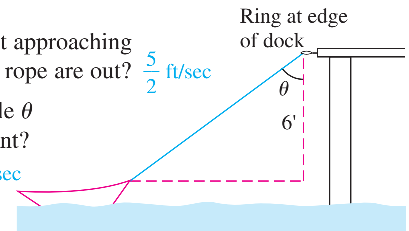
20. *Filling a Trough* A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min. How fast is the water level rising when it is 2 ft deep? $\frac{1}{16}$ ft/min



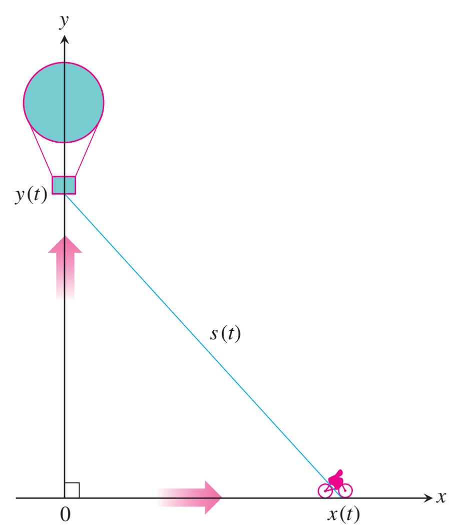
21. *Hauling in a Dinghy* A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

(a) How fast is the boat approaching the dock when 10 ft of rope are out? $\frac{5}{2}$ ft/sec

(b) At what rate is angle θ changing at that moment?
$-\frac{3}{20}$ radian/sec



22. *Rising Balloon* A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)? 11 ft/sec



In Exercises 23 and 24, a particle is moving along the curve $y = f(x)$.

23. Let $y = f(x) = \dfrac{10}{1 + x^2}$.
(a) $\frac{24}{5}$ cm/sec **(b)** 0 cm/sec **(c)** $-\frac{1200}{160{,}801} \approx -0.00746$ cm/sec

If $dx/dt = 3$ cm/sec, find dy/dt at the point where

(a) $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

24. Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find dy/dt at the point where

(a) $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.

(a) -46 cm/sec **(b)** 2 cm/sec **(c)** -88 cm/sec

25. *Particle Motion* A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination θ of theline joining the particle to the origin changing when $x = 3$? 1 radian/sec

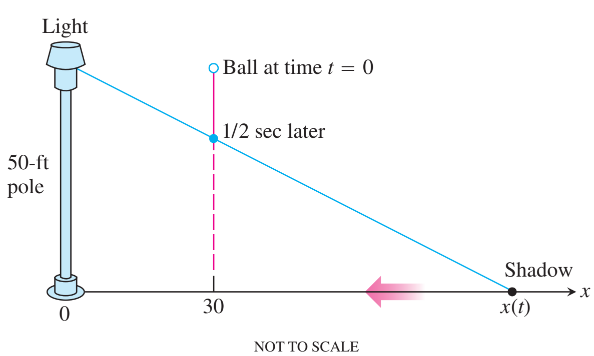
26. *Particle Motion* A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$? $\frac{2}{5}$ radian/sec

27. *Melting Ice* A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm? 1.6 cm²/min

28. *Particle Motion* A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)? -5 m/sec

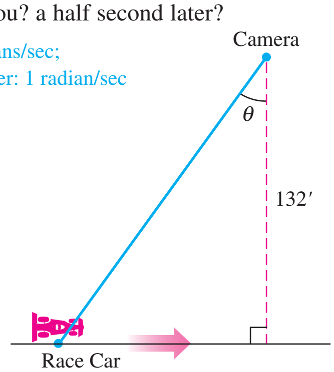
29. *Moving Shadow* A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light? -3 ft/sec

30. *Moving Shadow* A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown below. How fast is the ball's shadow moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.) -1500 ft/sec

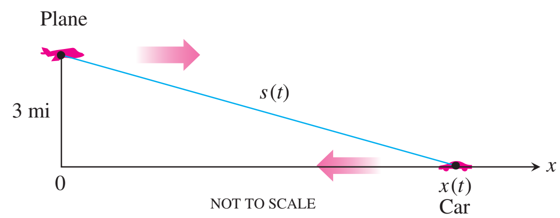


31. *Moving Race Car* You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle θ be changing when the car is right in front of you? a half second later?

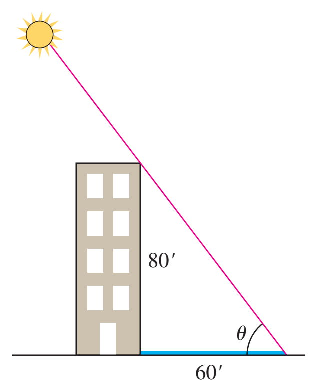
In front: 2 radians/sec;
Half second later: 1 radian/sec



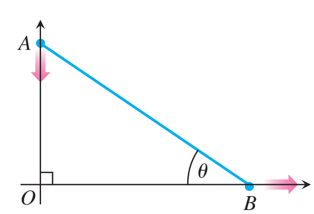
32. *Speed Trap* A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway. 80 mph



33. *Building's Shadow* On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.) 7.1 in./min



34. *Walkers* A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec and B moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree. -6 deg/sec



35. *Moving Ships* Two ships are steaming away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles? 29.5 knots

36. True. Since $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$, a constant $\frac{dr}{dt}$ results in a constant $\frac{dC}{dt}$. **37.** False. Since $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, the value of $\frac{dA}{dt}$ depends on r.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

36. True or False If the radius of a circle is expanding at a constant rate, then its circumference is increasing at a constant rate. Justify your answer.

37. True or False If the radius of a circle is expanding at a constant rate, then its area is increasing at a constant rate. Justify your answer.

38. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and each edge of the cube is increasing at 2 in./min, what is the length of each edge of the cube? A

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

39. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and the surface area of the cube is increasing at 12 in^2/min, what is the length of each edge of the cube? E

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

40. Multiple Choice A particle is moving around the unit circle (the circle of radius 1 centered at the origin). At the point (0.6, 0.8) the particle has horizontal velocity $dx/dt = 3$. What is its vertical velocity dy/dt at that point? C

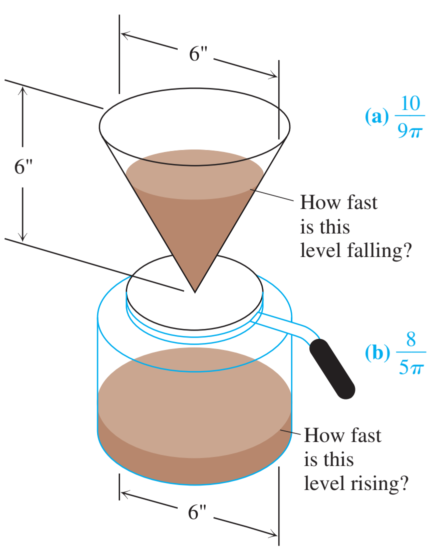
(A) -3.875 **(B)** -3.75 **(C)** -2.25 **(D)** 3.75 **(E)** 3.875

41. Multiple Choice A cylindrical rubber cord is stretched at a constant rate of 2 cm per second. Assuming its volume does not change, how fast is its radius shrinking when its length is 100 cm and its radius is 1 cm? B

(A) 0 cm/sec **(B)** 0.01 cm/sec **(C)** 0.02 cm/sec
(D) 2 cm/sec **(E)** 3.979 cm/sec

Explorations

42. *Making Coffee* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.



(a) $\frac{10}{9\pi} \approx 0.354$ in./min

(b) $\frac{8}{5\pi} \approx 0.509$ in./min

(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?

43. *Cost, Revenue, and Profit* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$. $\frac{dc}{dt} = 0.3$ $\frac{dr}{dt} = 0.9$ $\frac{dp}{dt} = 0.6$

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$. $\frac{dc}{dt} = -1.5625$ $\frac{dr}{dt} = 3.5$ $\frac{dp}{dt} = 5.0625$

44. Group Activity *Cardiac Output* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

$\frac{dy}{dt} = \frac{466}{1681} \approx 0.277 \text{ L/min}^2$

Extending the Ideas

45. *Motion along a Circle* A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express θ as a function of time t.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and π.

46. *Ferris Wheel* A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. [*Hint:* See Exercise 45.]

(b) At $t = 0$ the point P on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point P when $t = 5$ sec and $t = 8$ sec.

47. ***Industrial Production*** **(a)** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.) 9% per year

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

(b) Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate? Increasing at 1% per year

9. (a) $\frac{dA}{dt} = 14\text{ cm}^2\text{/sec}$ **(b)** $\frac{dP}{dt} = 0\text{ cm/sec}$

(c) $\frac{dD}{dt} = -\frac{14}{13}\text{ cm/sec}$

(d) The area is increasing, because its derivative is positive.
The perimeter is not changing, because its derivative is zero.
The diagonal length is decreasing, because its derivative is negative.

12. $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. But $S = 4\pi r^2$, so we are given that $\frac{dV}{dt} = kS = 4k\pi r^2$. Substituting, $4k\pi r^2 = 4\pi r^2 \frac{dr}{dt}$ which gives $\frac{dr}{dt} = k$.

Quick Quiz for AP* Preparation: Sections 4.4–4.6

You may use a graphing calculator to solve the following problems.

1. Multiple Choice If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, x_3, be if the first approximation is $x_1 = 1$? B

(A) 0.453 **(B)** 0.465 **(C)** 0.495 **(D)** 0.600 **(E)** 1.977

2. Multiple Choice The sides of a right triangle with legs x and y and hypotenuse z increase in such a way that $dz/dt = 1$ and $dx/dt = 3\,dy/dt$. At the instant when $x = 4$ and $y = 3$, what is dx/dt? B

(A) $\frac{1}{3}$ **(B)** 1 **(C)** 2 **(D)** $\sqrt{5}$ **(E)** 5

3. Multiple Choice An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection? A

(A) 57.60 **(B)** 57.88 **(C)** 59.20 **(D)** 60.00 **(E)** 67.40

4. Free Response **(a)** Approximate $\sqrt{26}$ by using the linearization of $y = \sqrt{x}$ at the point (25, 5). Show the computation that leads to your conclusion.

(b) Approximate $\sqrt{26}$ by using a first guess of 5 and one iteration of Newton's method to approximate the zero of $x^2 - 26$. Show the computation that leads to your conclusion.

(c) Approximate $\sqrt[3]{26}$ by using an appropriate linearization. Show the computation that leads to your conclusion.

Chapter 4 Key Terms

absolute change (p. 240)
absolute maximum value (p. 187)
absolute minimum value (p. 187)
antiderivative (p. 200)
antidifferentiation (p. 200)
arithmetic mean (p. 204)
average cost (p. 224)
center of linear approximation (p. 233)
concave down (p. 207)
concave up (p. 207)
concavity test (p. 208)
critical point (p. 190)
decreasing function (p. 198)
differential (p. 237)
differential estimate of change (p. 239)
differential of a function (p. 239)
extrema (p. 187)
Extreme Value Theorem (p. 188)
first derivative test (p. 205)
first derivative test for local extrema (p. 205)
geometric mean (p. 204)
global maximum value (p. 177)
global minimum value (p. 177)
increasing function (p. 198)
linear approximation (p. 233)
linearization (p. 233)
local linearity (p. 233)
local maximum value (p. 189)
local minimum value (p. 189)
logistic curve (p. 210)
logistic regression (p. 211)
marginal analysis (p. 223)
marginal cost and revenue (p. 223)
Mean Value Theorem (p. 196)
monotonic function (p. 198)
Newton's method (p. 235)
optimization (p. 219)
percentage change (p. 240)
point of inflection (p. 208)
profit (p. 223)
quadratic approximation (p. 245)
related rates (p. 246)
relative change (p. 240)
relative extrema (p. 189)
Rolle's Theorem (p. 196)
second derivative test
for local extrema (p. 211)
standard linear approximation (p. 233)

1. Maximum: $\frac{4\sqrt{6}}{9}$ at $x = \frac{4}{3}$; minimum: -4 at $x = -2$

Chapter 4 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1 and 2, use analytic methods to find the global extreme values of the function on the interval and state where they occur.

1. $y = x\sqrt{2-x}, \quad -2 \le x \le 2$

2. $y = x^3 - 9x^2 - 21x - 11, \quad -\infty < x < \infty$ No global extrema

In Exercises 3 and 4, use analytic methods. Find the intervals on which the function is See page 260.

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

3. $y = x^2 e^{1/x^2}$ See page 260.

4. $y = x\sqrt{4-x^2}$ See page 260.

In Exercises 5–16, find the intervals on which the function is

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

5. $y = 1 + x - x^2 - x^4$

6. $y = e^{x-1} - x$

7. $y = \dfrac{1}{\sqrt[4]{1-x^2}}$

8. $y = \dfrac{x}{x^3 - 1}$

9. $y = \cos^{-1} x$

10. $y = \dfrac{x}{x^2 + 2x + 3}$

11. $y = \ln|x|, \quad -2 \le x \le 2, \quad x \ne 0$

12. $y = \sin 3x + \cos 4x, \quad 0 \le x \le 2\pi$

13. $y = \begin{cases} e^{-x}, & x \le 0 \\ 4x - x^3, & x > 0 \end{cases}$

14. $y = -x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2$

15. $y = x^{4/5}(2 - x)$

16. $y = \dfrac{5 - 4x + 4x^2 - x^3}{x - 2}$

In Exercises 17 and 18, use the derivative of the function $y = f(x)$ to find the points at which f has a

(a) local maximum, **(b)** local minimum, or

(c) point of inflection.

17. $y' = 6(x+1)(x-2)^2$
(a) None **(b)** At $x = -1$ **(c)** At $x = 0$ and $x = 2$

18. $y' = 6(x+1)(x-2)$
(a) At $x = -1$ **(b)** At $x = 2$ **(c)** At $x = \frac{1}{2}$

In Exercises 19–22, find all possible functions with the given derivative.

19. $f'(x) = x^{-5} + e^{-x}$
$f(x) = -\frac{1}{4}x^{-4} - e^{-x} + C$

20. $f'(x) = \sec x \tan x$
$f(x) = \sec x + C$

21. $f'(x) = \dfrac{2}{x} + x^2 + 1, \quad x > 0$
$f(x) = 2\ln x + \frac{1}{3}x^3 + x + C$

22. $f'(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
$f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

In Exercises 23 and 24, find the function with the given derivative whose graph passes through the point P.

23. $f'(x) = \sin x + \cos x, \quad P(\pi, 3)$ $f(x) = -\cos x + \sin x + 2$

24. $f'(x) = x^{1/3} + x^2 + x + 1, \quad P(1, 0)$
$f(x) = \frac{3}{4}x^{4/3} + \frac{x^3}{3} + \frac{x^2}{2} + x - \frac{31}{12}$

In Exercises 25 and 26, the velocity v or acceleration a of a particle is given. Find the particle's position s at time t.

25. $v = 9.8t + 5, \quad s = 10 \quad \text{when} \quad t = 0$ $s(t) = 4.9t^2 + 5t + 10$

26. $a = 32, \quad v = 20 \quad \text{and} \quad s = 5 \quad \text{when} \quad t = 0$ $s(t) = 16t^2 + 20t + 5$

In Exercises 27–30, find the linearization $L(x)$ of $f(x)$ at $x = a$.

27. $f(x) = \tan x, \quad a = -\pi/4$
See page 260.

28. $f(x) = \sec x, \quad a = \pi/4$
See page 260.

29. $f(x) = \dfrac{1}{1 + \tan x}, \quad a = 0$
$L(x) = -x + 1$

30. $f(x) = e^x + \sin x, \quad a = 0$
$L(x) = 2x + 1$

In Exercises 31–34, use the graph to answer the questions.

31. Identify any global extreme values of f and the values of x at which they occur. Global minimum value of $\frac{1}{2}$ at $x = 2$

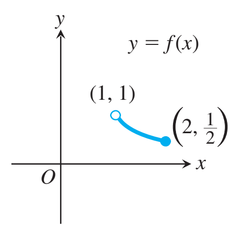


Figure for Exercise 31

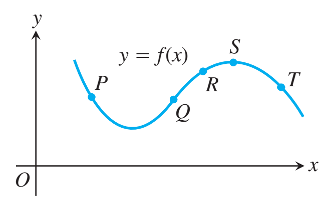


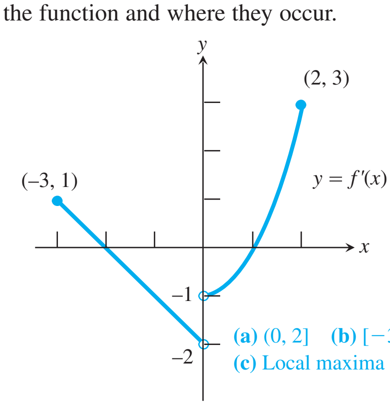
Figure for Exercise 32

32. At which of the five points on the graph of $y = f(x)$ shown here

(a) are y' and y'' both negative? T

(b) is y' negative and y'' positive? P

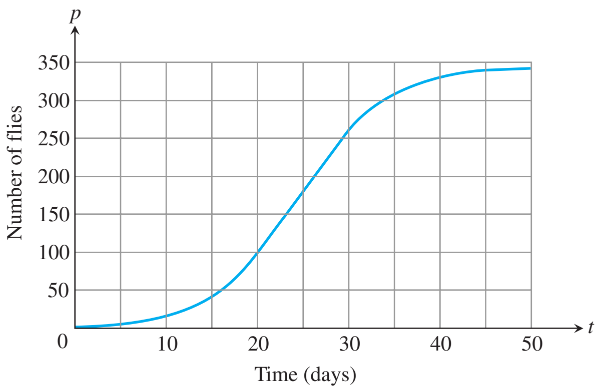
33. Estimate the intervals on which the function $y = f(x)$ is **(a)** increasing; **(b)** decreasing. **(c)** Estimate any local extreme values of the function and where they occur.



(a) $(0, 2]$ **(b)** $[-3, 0)$
(c) Local maxima at $(-3, 1)$ and $(2, 3)$

34. Here is the graph of the fruit fly population from Section 2.4, Example 2. On approximately what day did the population's growth rate change from increasing to decreasing? The 24th day

37. (a) $f(x)$ is continuous on $[0.5, 3]$ and differentiable on $(0.5, 3)$.



35. ***Connecting f and f′*** The graph of f' is shown in Exercise 33. Sketch a possible graph of f given that it is continuous with domain $[-3, 2]$ and $f(-3) = 0$.

36. ***Connecting f, f′, and f″*** The function f is continuous on $[0, 3]$ and satisfies the following.

x	0	1	2	3
f	0	-2	0	3
f'	-3	0	does not exist	4
f''	0	1	does not exist	0

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	$-$	$-$	$+$
f'	$-$	$+$	$+$
f''	$+$	$+$	$+$

(a) Find the absolute extrema of f and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of f.

37. ***Mean Value Theorem*** Let $f(x) = x \ln x$.

(a) Writing to Learn Show that f satisfies the hypotheses of the Mean Value Theorem on the interval $[a, b] = [0.5, 3]$. See above.

(b) Find the value(s) of c in (a, b) for which $c \approx 1.579$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(c) Write an equation for the secant line AB where $A = (a, f(a))$ and $B = (b, f(b))$. $y \approx 1.457x - 1.075$

(d) Write an equation for the tangent line that is parallel to the secant line AB. $y \approx 1.457x - 1.579$

38. ***Motion along a Line*** A particle is moving along a line with position function $s(t) = 3 + 4t - 3t^2 - t^3$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \ge 0$. See page 260.

39. ***Approximating Functions*** Let f be a function with $f'(x) = \sin x^2$ and $f(0) = -1$. See page 260.

(a) Find the linearization of f at $x = 0$.

(b) Approximate the value of f at $x = 0.1$.

(c) Writing to Learn Is the actual value of f at $x = 0.1$ greater than or less than the approximation in (b)?

40. ***Differentials*** Let $y = x^2 e^{-x}$. Find **(a)** dy and **(b)** evaluate dy for $x = 1$ and $dx = 0.01$. **(a)** $dy = (2x - x^2)e^{-x}\,dx$ **(b)** $dy \approx 0.00368$

41. Table 4.5 shows the growth of the population of Tennessee from the 1850 census to the 1910 census. The table gives the population growth beyond the baseline number from the 1840 census, which was 829,210.

Table 4.5 Population Growth of Tennessee

Years since 1840	Growth Beyond 1840 Population
10	173,507
20	280,591
30	429,310
40	713,149
50	938,308
60	1,191,406
70	1,355,579

Source: Bureau of the Census, U.S. Chamber of Commerce

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Tennessee population in the 1920 census. Be sure to add the baseline 1840 number. (The actual 1920 census value was 2,337,885.)

(d) In what year during the period was the Tennessee population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Tennessee in the long run?

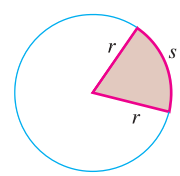
(f) Writing to Learn In fact, the population of Tennessee had already passed the long-run value predicted by this regression curve by 1930. By 2000 it had surpassed the prediction by more than 3 million people! What historical circumstances could have made the early regression so unreliable?

42. ***Newton's Method*** Use Newton's method to estimate all real solutions to $2 \cos x - \sqrt{1 + x} = 0$. State your answers accurate to 6 decimal places. $x \approx 0.828361$

43. ***Rocket Launch*** A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later? 1200 m/sec

44. ***Launching on Mars*** The acceleration of gravity near the surface of Mars is 3.72 m/sec^2. If a rock is blasted straight up from the surface with an initial velocity of 93 m/sec (about 208 mph), how high does it go? 1162.5 m

45. ***Area of Sector*** If the perimeter of the circular sector shown here is fixed at 100 ft, what values of r and s will give the sector the greatest area? $r = 25$ ft and $s = 50$ ft

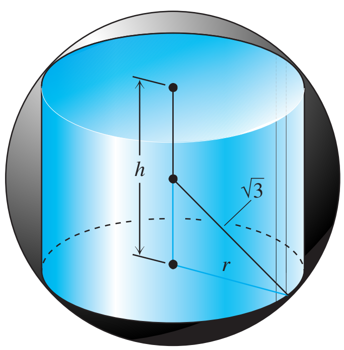


46. ***Area of Triangle*** An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have. 54 square units

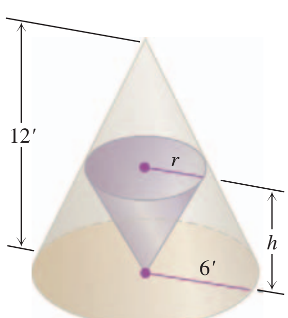
47. ***Storage Bin*** Find the dimensions of the largest open-top storage bin with a square base and vertical sides that can be made from 108 ft^2 of sheet steel. (Neglect the thickness of the steel and assume that there is no waste.) Base is 6 ft by 6 ft, height = 3 ft

48. ***Designing a Vat*** You are to design an open-top rectangular stainless-steel vat. It is to have a square base and a volume of 32 ft^3; to be welded from quarter-inch plate, and weigh no more than necessary. What dimensions do you recommend? Base is 4 ft by 4 ft, height = 2 ft

49. ***Inscribing a Cylinder*** Find the height and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$ as described in the figure. Height = 2, radius = $\sqrt{2}$



50. ***Cone in a Cone*** The figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume? $r = h = 4$ ft

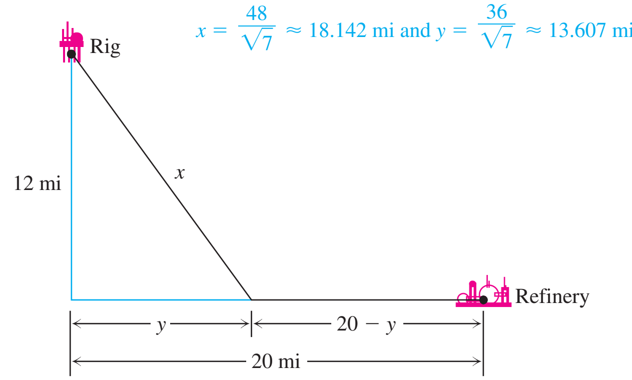


51. ***Box with Lid*** Repeat Exercise 18 of Section 4.4 but this time remove the two equal squares from the corners of a 15-in. side.

52. ***Inscribing a Rectangle*** A rectangle is inscribed under one arch of $y = 8 \cos (0.3x)$ with its base on the x-axis and its upper two vertices on the curve symmetric about the y-axis. What is the largest area the rectangle can have? 29.925 square units

53. ***Oil Refinery*** A drilling rig 12 mi offshore is to be connected by a pipe to a refinery onshore, 20 mi down the coast from the rig as shown in the figure. If underwater pipe costs \$40,000 per mile and land-based pipe costs \$30,000 per mile, what values of x and y give the least expensive connection?

$x = \dfrac{48}{\sqrt{7}} \approx 18.142$ mi and $y = \dfrac{36}{\sqrt{7}} \approx 13.607$ mi



54. ***Designing an Athletic Field*** An athletic field is to be built in the shape of a rectangle x units long capped by semicircular regions of radius r at the two ends. The field is to be bounded by a 400-m running track. What values of x and r will give the rectangle the largest possible area? $x = 100$ m and $r = \dfrac{100}{\pi}$ m

55. ***Manufacturing Tires*** Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and 276 grade A and 553 grade B tires

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

56. ***Particle Motion*** The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos (t + \pi/4)$.

(a) What is the farthest apart the particles ever get? 0.765 units

(b) When do the particles collide? When $t = \dfrac{7\pi}{8} \approx 2.749$ (plus multiples of π if they keep going)

57. ***Open-top Box*** An open-top rectangular box is constructed from a 10- by 16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

58. ***Changing Area*** The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m? -40 m^2/sec

57. Dimensions: base is 6 in. by 12 in., height = 2 in.; maximum volume = 144 in^3

59. ***Particle Motion*** The coordinates of a particle moving in the plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)? 5 m/sec

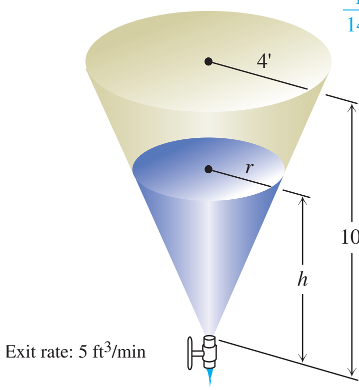
60. ***Changing Cube*** The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the edges changing at that instant? Increasing 1 cm/min

61. ***Particle Motion*** A point moves smoothly along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the constant rate of 11 units per second. Find dx/dt when $x = 3$. $\frac{dx}{dt} = 4$ units/second

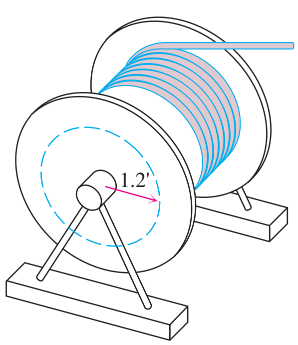
62. ***Draining Water*** Water drains from the conical tank shown in the figure at the rate of 5 ft^3/min.

(a) What is the relation between the variables h and r? **(a)** $h = \frac{5r}{2}$

(b) How fast is the water level dropping when $h = 6$ ft? $\frac{125}{144\pi} \approx 0.276$ ft/min

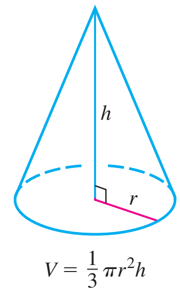


63. ***Stringing Telephone Cable*** As telephone cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius as suggested in the figure. If the truck pulling the cable moves at a constant rate of 6 ft/sec, use the equation $s = r\theta$ to find how fast (in rad/sec) the spool is turning when the layer of radius 1.2 ft is being unwound. 5 radians/sec



64. ***Throwing Dirt*** You sling a shovelful of dirt up from the bottom of a 17-ft hole with an initial velocity of 32 ft/sec. Is that enough speed to get the dirt out of the hole, or had you better duck? Not enough speed. Duck!

65. ***Estimating Change*** Write a formula that estimates the change that occurs in the volume of a right circular cone (see figure) when the radius changes from a to $a + dr$ and the height does not change. $dV \approx \frac{2\pi ah}{3}\, dr$



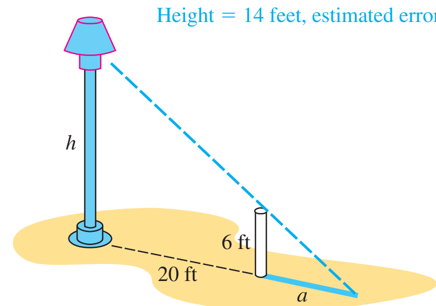
66. ***Controlling Error***

(a) How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%? Within 1%

(b) Suppose the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement. Within 3%

67. ***Compounding Error*** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of **(a)** the radius, **(b)** the surface area, and **(c)** the volume.

68. ***Finding Height*** To find the height of a lamppost (see figure), you stand a 6-ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$, and estimate the possible error in the result. Height = 14 feet, estimated error = $\pm\frac{2}{45}$ feet



67. **(a)** Within 4% **(b)** Within 8% **(c)** Within 12%

69. *Decreasing Function* Show that the function $y = \sin^2 x - 3x$ decreases on every interval in its domain.

AP Examination Preparation

You should solve the following problems without using a calculator.

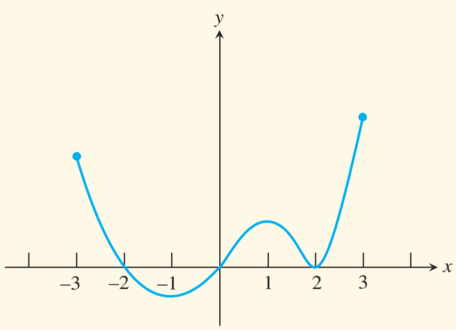
70. The accompanying figure shows the graph of the derivative of a function f. The domain of f is the closed interval $[-3, 3]$.

(a) For what values of x in the open interval $(-3, 3)$ does f have a relative maximum? Justify your answer.

(b) For what values of x in the open interval $(-3, 3)$ does f have a relative minimum? Justify your answer.

(c) For what values of x is the graph of f concave up? Justify your answer.

(d) Suppose $f(-3) = 0$. Sketch a possible graph of f on the domain $[-3, 3]$.



71. The volume V of a cone $(V = \frac{1}{3}\pi r^2 h)$ is increasing at the rate of 4π cubic inches per second. At the instant when the radius of the cone is 2 inches, its volume is 8π cubic inches and the radius is increasing at 1/3 inch per second.

(a) At the instant when the radius of the cone is 2 inches, what is the rate of change of the area of its base?

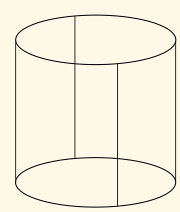
(b) At the instant when the radius of the cone is 2 inches, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 2 inches, what is the instantaneous rate of change of the area of its base with respect to its height h?

72. A piece of wire 60 inches long is cut into six sections, two of length a and four of length b. Each of the two sections of length a is bent into the form of a circle and the circles are then joined by the four sections of length b to make a frame for a model of a right circular cylinder, as shown in the accompanying figure.

(a) Find the values of a and b that will make the cylinder of maximum volume.

(b) Use differential calculus to justify your answer in part (a).



3. (a) $[-1, 0)$ and $[1, \infty)$
(b) $(-\infty, -1]$ and $(0, 1]$
(c) $(-\infty, 0)$ and $(0, \infty)$
(d) None
(e) Local minima at $(1, e)$ and $(-1, e)$
(f) None

4. (a) $[-\sqrt{2}, \sqrt{2}]$
(b) $[-2, -\sqrt{2}]$ and $[\sqrt{2}, 2]$
(c) $(-2, 0)$ **(d)** $(0, 2)$
(e) Local max: $(-2, 0)$ and $(\sqrt{2}, 2)$;
local min: $(2, 0)$ and $(-\sqrt{2}, -2)$

27. $L(x) = 2x + \frac{\pi}{2} - 1$

28. $L(x) = \sqrt{2}x - \frac{\pi\sqrt{2}}{4} + \sqrt{2}$

38. (a) $v(t) = -3t^2 - 6t + 4$
(b) $a(t) = -6t - 6$
(c) The particle starts at position 3 moving in the positive direction, but decelerating. At approximately $t = 0.528$, it reaches position 4.128 and changes direction, beginning to move in the negative direction. After that, it continues to accelerate while moving in the negative direction.

39. (a) $L(x) = -1$
(b) Using the linearization, $f(0.1) \approx -1$
(c) Greater than the approximation in (b), since $f'(x)$ is actually positive over the interval $(0, 0.1)$ and the estimate is based on the derivative being 0.

69. $\frac{dy}{dx} = 2 \sin x \cos x - 3$.
Since $\sin x$ and $\cos x$ are both between 1 and -1,
$2 \sin x \cos x$ is never greater than 2, and therefore $\frac{dy}{dx} \le 2 - 3 = -1$ for all values of x.

71. The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 4π cubic inches per second. At the instant when the radius of the cone is 2 inches, its volume is 8π cubic inches and the radius is increasing at 1/3 inch per second.

(a) $A = \pi r^2$, so $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(2)\left(\frac{1}{3}\right) = \frac{4\pi}{3}$ in²/sec.

(b) $V = \frac{1}{3}\pi r^2 h$, so $\frac{dV}{dt} = \frac{1}{3}\left(2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}\right)$. Plugging in the known values, we have $4\pi = \frac{1}{3}\left(2\pi \cdot 2 \cdot \frac{1}{3} \cdot 6 + \pi \cdot 2^2 \cdot \frac{dh}{dt}\right)$. From this we get $\frac{dh}{dt} = 1$ in/sec.

(c) $\frac{dA}{dh} = \frac{dA/dt}{dh/dt} = \frac{4\pi/3}{1} = \frac{4\pi}{3}$ in²/in.