Finding Limits Using Properties of Limits
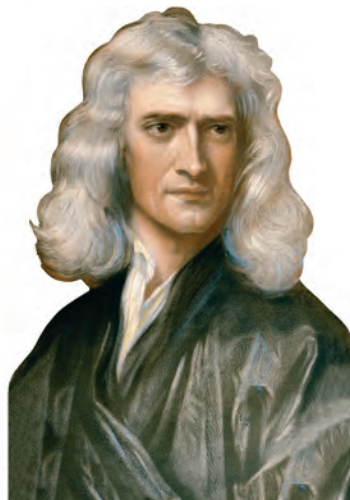
Objectives

1. Find limits of constant functions and the identity function.
2. Find limits using properties of limits.
3. Find one-sided limits using properties of limits.
4. Find limits of fractional expressions in which the limit of the denominator is zero.

Isaac Newton   Gottfried Leibniz

Calculus was invented independently by British mathematician Isaac Newton (1642–1727) and German mathematician Gottfried Leibniz (1646–1716). Although Newton stated that limits were the basic concept in calculus, neither he nor Leibniz was able to express the idea of a limit in a precise mathematical fashion. In essence, Newton and Leibniz developed calculus into a powerful tool even though they could not fully understand why the tool worked.

  A great triumph of calculus came with the work of German mathematician Karl Weierstrass (1815–1897). Weierstrass provided a precise definition of $\lim\limits_{x \to a} f(x) = L$, placing calculus on a sound footing almost two hundred years after its invention. The properties of limits presented in this section are theorems that you will prove in calculus using this definition. In this section, you will learn to apply these properties to find limits.

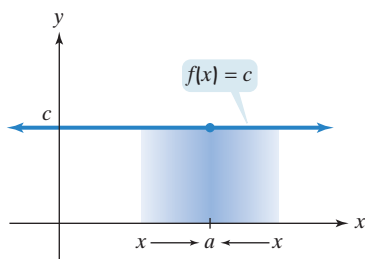① Find limits of constant functions and the identity function.

y

$f(x) = c$

c

$x \longrightarrow a \longleftarrow x$

x

Figure 11.8 The graph of the constant function $f(x) = c$. No matter how close x is to a, the corresponding value of $f(x)$ is c.

Limits Involving Constant Functions and the Identity Function

We frequently encounter the constant function, $f(x) = c$, and the identity function, $f(x) = x$. **Figure 11.8** shows the graph of the constant function. The graph is a horizontal line. What does this mean about the limit as x approaches a? Regardless of how close x is to a, the corresponding value of $f(x)$ is c. Thus, if $f(x) = c$, then $\lim\limits_{x \to a} f(x) = c$.

> **Limit of a Constant Function**
>
> For the constant function $f(x) = c$,
>
> $$\lim_{x \to a} f(x) = \lim_{x \to a} c = c,$$
>
> where a is any number. In words, regardless of what number x is approaching, the limit of any constant is that constant.

EXAMPLE 1 **Finding Limits of Constant Functions**

Find the following limits:

 a. $\lim\limits_{x \to 4} 7$ **b.** $\lim\limits_{x \to 0} (-5)$.

Solution   Regardless of what number x is approaching, the limit of any constant is that constant: $\lim\limits_{x \to a} c = c$. Using this formula, we find the given limits.

a. $\lim\limits_{x \to 4} 7 = 7$ **b.** $\lim\limits_{x \to 0} (-5) = -5$ ●

⊘ **Check Point** **1**   Find the following limits:

   a. $\lim\limits_{x \to 8} 11$ **b.** $\lim\limits_{x \to 0}(-9)$.

The graph of the identity function, $f(x) = x$, is shown in **Figure 11.9**. Each input for this function is associated with an identical output. What does this mean about the limit as x approaches a? For any value of a, as x gets closer to a, the corresponding value of $f(x)$ is just as close to a. Thus, if $f(x) = x$, then $\lim\limits_{x \to a} f(x) = a$.
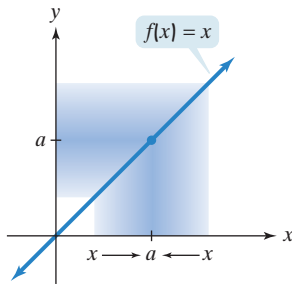

Figure 11.9   The graph of the identity function $f(x) = x$. No matter how close x is to a, the corresponding value of $f(x)$ is just as close to a.

> **Limit of the Identity Function**
>
> For the identity function $f(x) = x$,
>
> $$\lim_{x \to a} f(x) = \lim_{x \to a} x = a,$$
>
> where a is any number. In words, the limit of x as x approaches any number is that number.

EXAMPLE 2   **Finding Limits of the Identity Function**

Find the following limits:

   a. $\lim\limits_{x \to 7} x$ **b.** $\lim\limits_{x \to -\pi} x$.

Solution   We use the formula $\lim\limits_{x \to a} x = a$. The number that x is approaching is also the limit.

   a. $\lim\limits_{x \to 7} x = 7$ **b.** $\lim\limits_{x \to -\pi} x = -\pi$ ●

⊘ **Check Point** **2**   Find the following limits:

   a. $\lim\limits_{x \to 19} x$ **b.** $\lim\limits_{x \to -\sqrt{2}} x$.

2 Find limits using properties of limits.

Properties of Limits

How do we find the limit of a sum, such as

$$\lim_{x \to 5}(x + 7)?$$

We find the limit of each function in the sum:

$$\lim_{x \to 5} x = 5 \qquad \text{and} \qquad \lim_{x \to 5} 7 = 7.$$

Use the formula $\lim\limits_{x \to a} x = a$.   Use the formula $\lim\limits_{x \to a} c = c$.

Then we add each of these limits. Thus,

$$\lim_{x \to 5}(x + 7) = \lim_{x \to 5} x + \lim_{x \to 5} 7 = 5 + 7 = 12.$$

This is an application of a limit property involving the limit of a sum.

The Limit of a Sum

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M.$$

In words, the limit of the sum of two functions equals the sum of their limits.

EXAMPLE 3 Finding the Limit of a Sum

Find: $\lim\limits_{x \to -4} (x + 9)$.

Solution The two functions in this limit problem are $f(x) = x$ and $g(x) = 9$. We seek the limit of the sum of these functions.

$$\lim_{x \to -4} (x + 9) = \lim_{x \to -4} x + \lim_{x \to -4} 9 \qquad \text{The limit of a sum is the sum of the limits.}$$

$$= -4 + 9 \qquad \lim_{x \to a} x = a \text{ and } \lim_{x \to a} c = c.$$

$$= 5$$

✓ **Check Point 3** Find: $\lim\limits_{x \to -3} (x + 16)$.

In calculus, you will prove the following property involving the limit of the difference of two functions:

The Limit of a Difference

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M.$$

In words, the limit of the difference of two functions equals the difference of their limits.

EXAMPLE 4 Finding the Limit of a Difference

Find: $\lim\limits_{x \to 5} (12 - x)$.

Solution The two functions in this limit problem are $f(x) = 12$ and $g(x) = x$. We seek the limit of the difference of these functions.

$$\lim_{x \to 5} (12 - x) = \lim_{x \to 5} 12 - \lim_{x \to 5} x \qquad \text{The limit of a difference is the difference of the limits.}$$

$$= 12 - 5 \qquad \lim_{x \to a} c = c \text{ and } \lim_{x \to a} x = a.$$

$$= 7$$

✓ **Check Point 4** Find: $\lim\limits_{x \to 14} (19 - x)$.

Now we consider a property that will enable you to find the limit of the product of two functions.

The Limit of a Product

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then

$$\lim\limits_{x \to a}[f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = LM.$$

In words, the limit of the product of two functions equals the product of their limits.

EXAMPLE 5 **Finding the Limit of a Product**

Find: $\lim\limits_{x \to 5}(-6x)$.

Solution The two functions in this limit problem are $f(x) = -6$ and $g(x) = x$. We seek the limit of the product of these functions.

$$\lim\limits_{x \to 5}(-6x) = \lim\limits_{x \to 5}(-6) \cdot \lim\limits_{x \to 5} x \qquad \text{The limit of a product is the product of the limits.}$$
$$= -6 \cdot 5 \qquad \qquad \lim\limits_{x \to a} c = c \text{ and } \lim\limits_{x \to a} x = a.$$
$$= -30$$

○ **Check Point 5** Find: $\lim\limits_{x \to 7}(-10x)$.

EXAMPLE 6 **Finding Limits Using Properties of Limits**

Find the following limits:

a. $\lim\limits_{x \to -3}(7x - 4)$ **b.** $\lim\limits_{x \to 5} 6x^2$.

Solution

a. $\lim\limits_{x \to -3}(7x - 4) = \lim\limits_{x \to -3}(7x) - \lim\limits_{x \to -3} 4$ The limit of a difference is the difference of the limits.

$$= \lim\limits_{x \to -3} 7 \cdot \lim\limits_{x \to -3} x - \lim\limits_{x \to -3} 4 \qquad \text{The limit of a product is the product of the limits.}$$
$$= 7(-3) - 4 \qquad \qquad \lim\limits_{x \to a} c = c \text{ and } \lim\limits_{x \to a} x = a.$$
$$= -21 - 4$$
$$= -25$$

b. $\lim\limits_{x \to 5} 6x^2 = \lim\limits_{x \to 5} 6 \cdot \lim\limits_{x \to 5} x^2$ The limit of a product is the product of the limits.

$$= 6 \cdot \lim\limits_{x \to 5}(x \cdot x) \qquad \qquad \lim\limits_{x \to a} c = c$$
$$= 6 \cdot \lim\limits_{x \to 5} x \cdot \lim\limits_{x \to 5} x \qquad \text{The limit of a product is the product of the limits.}$$
$$= 6 \cdot 5 \cdot 5 \qquad \qquad \lim\limits_{x \to a} x = a$$
$$= 150$$

○ **Check Point 6** Find the following limits:

a. $\lim\limits_{x \to -5}(3x - 7)$ **b.** $\lim\limits_{x \to 3} 8x^2$.

The procedure used to find $\lim\limits_{x \to 5} 6x^2$ in Example 6(b) can be used to determine the limit of any monomial function in the form $f(x) = b_n x^n$, where n is a positive integer and b_n is a constant.

$$\lim_{x \to a} b_n x^n = \lim_{x \to a} b_n \cdot \lim_{x \to a} x^n \qquad \text{The limit of a product is the product of the limits.}$$

$$= b_n \cdot \lim_{x \to a} (\underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}) \qquad \lim_{x \to a} c = c$$

By definition, x^n contains n factors of x.

$$= b_n \cdot \underbrace{\lim_{x \to a} x \cdot \lim_{x \to a} x \cdot \lim_{x \to a} x \cdot \cdots \cdot \lim_{x \to a} x} \qquad \text{The limit of the product containing } n \text{ factors is the product of the limits.}$$

There are n factors of $\lim\limits_{x \to a} x$.

$$= b_n \cdot \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a} \qquad \lim_{x \to a} x = a$$

There are n factors of a.

$$= b_n a^n$$

This is the monomial function $f(x) = b_n x^n$ evaluated at a.

Limit of a Monomial

If n is a positive integer and b_n is a constant, then

$$\lim_{x \to a} b_n x^n = b_n a^n$$

for any number a. In words, the limit of a monomial as x approaches a is the monomial evaluated at a.

EXAMPLE 7 **Finding the Limit of a Monomial**

Find: $\lim\limits_{x \to 2} (-6x^4)$.

Solution The limit of the monomial $-6x^4$ as x approaches 2 is the monomial evaluated at 2. Thus, we find the limit by substituting 2 for x.

$$\lim_{x \to 2} (-6x^4) = -6 \cdot 2^4 = -6 \cdot 16 = -96 \qquad \bullet$$

✓ Check Point **7** Find: $\lim\limits_{x \to 2} (-7x^3)$.

How do we find the limit of a polynomial function

$$f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

as x approaches a? A polynomial is a sum of monomials. Thus, the limit of a polynomial is the sum of the limits of its monomials.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0) \qquad f \text{ is a polynomial function.}$$

$$= \lim_{x \to a} b_n x^n + \lim_{x \to a} b_{n-1} x^{n-1} + \cdots + \lim_{x \to a} b_1 x + \lim_{x \to a} b_0 \qquad \text{The limit of a sum is the sum of the limits.}$$

$$= b_n a^n + b_{n-1} a^{n-1} + \cdots + b_1 a + b_0 \qquad \text{Find limits by evaluating monomials at } a. \text{ Find the last limit in the sum using } \lim_{x \to a} c = c \text{ with } c = b_0.$$

This is the polynomial function
$f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$
evaluated at a.

$$= f(a)$$

> ### Limit of a Polynomial
> If f is a polynomial function, then
> $$\lim_{x \to a} f(x) = f(a)$$
> for any number a. In words, the limit of a polynomial as x approaches a is the polynomial evaluated at a.

(EXAMPLE 8) **Finding the Limit of a Polynomial**

Find: $\lim\limits_{x \to 3}(4x^3 + 2x^2 - 6x + 5)$.

Solution   The limit of the polynomial $4x^3 + 2x^2 - 6x + 5$ as x approaches 3 is the polynomial evaluated at 3. Thus, we find the limit by substituting 3 for x.

$$\lim_{x \to 3}(4x^3 + 2x^2 - 6x + 5)$$
$$= 4 \cdot 3^3 + 2 \cdot 3^2 - 6 \cdot 3 + 5$$
$$= 4 \cdot 27 + 2 \cdot 9 - 6 \cdot 3 + 5$$
$$= 108 + 18 - 18 + 5$$
$$= 113$$

●

Check Point 8 Find: $\lim\limits_{x \to 2}(7x^3 + 3x^2 - 5x + 3)$.

A linear function, $f(x) = mx + b$, is a polynomial function of degree one. This means that the limit of a linear function as x approaches a is the linear function evaluated at a:
$$\lim_{x \to a}(mx + b) = ma + b.$$
For example,
$$\lim_{x \to 4}(3x - 7) = 3 \cdot 4 - 7 = 12 - 7 = 5.$$

The next limit property involves the limit of a function to a power, such as
$$\lim_{x \to 2}(x^2 + 2x - 3)^4.$$
To find such a limit, first find $\lim\limits_{x \to 2}(x^2 + 2x - 3)$:
$$\lim_{x \to 2}(x^2 + 2x - 3) = 2^2 + 2 \cdot 2 - 3 = 4 + 4 - 3 = 5.$$
The limit that we seek is found by taking this limit, 5, and raising it to the fourth power. Thus,
$$\lim_{x \to 2}(x^2 + 2x - 3)^4 = \left[\lim_{x \to 2}(x^2 + 2x - 3)\right]^4 = 5^4 = 625.$$

> ### The Limit of a Power
> If $\lim\limits_{x \to a} f(x) = L$ and n is a positive integer, then
> $$\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = L^n.$$
> In words, the limit of a function to a power is found by taking the limit of the function and then raising this limit to the power.

Calculus in Japan

Mathematics was developed simultaneously in various cultures in all parts of the world. Seki Kowa, a 17th-century Japanese mathematician, is credited with the invention of calculus in Japan. The illustration shown below was drawn in 1670 by a pupil of Seki Kowa. It measures the circle's area with a series of rectangles.
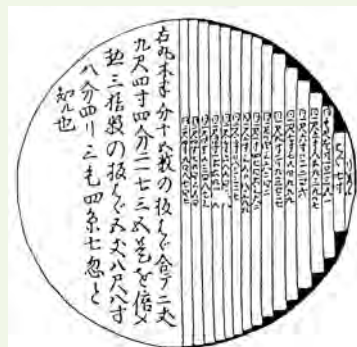
EXAMPLE 9 Finding the Limit of a Power

Find: $\lim\limits_{x \to 5}(2x - 7)^3$.

Solution The limit of the linear function $f(x) = 2x - 7$ as x approaches 5 is the linear function evaluated at 5. Because this function is raised to the third power, the limit that we seek is the limit of the linear function raised to the third power.

$$\lim\limits_{x \to 5}(2x - 7)^3 = \left[\lim\limits_{x \to 5}(2x - 7)\right]^3 = (2 \cdot 5 - 7)^3 = 3^3 = 27$$ ●

🗸 Check Point **9** Find: $\lim\limits_{x \to 4}(3x - 5)^3$.

How do we find the limit of a root? Recall that if $\sqrt[n]{L}$ represents a real number and $n \geq 2$, then

$$\sqrt[n]{L} = L^{1/n}.$$

Because a root is a power, we find the limit of a root using a similar procedure for finding the limit of a power.

The Limit of a Root

If $\lim\limits_{x \to a} f(x) = L$ and n is a positive integer greater than or equal to 2, then

$$\lim\limits_{x \to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$$

provided that all roots represent real numbers. In words, the limit of the nth root of a function is found by taking the limit of the function and then taking the nth root of this limit.

EXAMPLE 10 Finding the Limit of a Root

Find: $\lim\limits_{x \to -2}\sqrt{4x^2 + 5}$.

Solution The limit of the quadratic (polynomial) function $f(x) = 4x^2 + 5$ as x approaches -2 is the function evaluated at -2. Because we have the square root of this function, the limit that we seek is the square root of the limit of the quadratic function.

$$\lim\limits_{x \to -2}\sqrt{4x^2 + 5} = \sqrt{\lim\limits_{x \to -2}(4x^2 + 5)} = \sqrt{4(-2)^2 + 5} = \sqrt{16 + 5} = \sqrt{21}$$ ●

🗸 Check Point **10** Find: $\lim\limits_{x \to -1}\sqrt{6x^2 - 4}$.

We have considered limits of sums, differences, products, and roots. We conclude with a property that deals with the limit of a quotient.

The Limit of a Quotient

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, $M \neq 0$, then

$$\lim\limits_{x \to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}, M \neq 0.$$

In words, the limit of the quotient of two functions equals the quotient of their limits, **provided that the limit of the denominator is not zero**.

Before possibly applying the quotient property, begin by finding the limit of the denominator. If this limit is not zero, you can apply the quotient property. If this limit is zero, the quotient property cannot be used.

EXAMPLE 11 **Finding the Limit of a Quotient**

Find: $\displaystyle\lim_{x \to 1} \frac{x^3 - 3x^2 + 7}{2x - 5}$.

Solution The two functions in this limit problem are $f(x) = x^3 - 3x^2 + 7$ and $g(x) = 2x - 5$. We seek the limit of the quotient of these functions. Can we use the quotient property for limits? We answer the question by finding the limit of the denominator, $g(x)$.

$$\lim_{x \to 1} (2x - 5) = 2 \cdot 1 - 5 = -3$$

Because the limit of the denominator is not zero, we can apply the quotient property for limits. The limit of the quotient is the quotient of the limits.

$$\lim_{x \to 1} \frac{x^3 - 3x^2 + 7}{2x - 5} = \frac{\lim_{x \to 1}(x^3 - 3x^2 + 7)}{\lim_{x \to 1}(2x - 5)} = \frac{1^3 - 3 \cdot 1^2 + 7}{2 \cdot 1 - 5} = \frac{5}{-3} = -\frac{5}{3} \quad \bullet$$

✓ **Check Point 11** Find: $\displaystyle\lim_{x \to 2} \frac{x^2 - 4x + 1}{3x - 5}$.

We've considered a number of limit properties. Let's take a moment to summarize these properties.

Properties of Limits

Formulas for Finding Limits

1. $\displaystyle\lim_{x \to a} c = c$

2. $\displaystyle\lim_{x \to a} x = a$

3. If f is a polynomial (linear, quadratic, cubic, etc.) function, $\displaystyle\lim_{x \to a} f(x) = f(a)$.

Limits of Sums, Differences, Products, Powers, Roots, and Quotients

If $\displaystyle\lim_{x \to a} f(x) = L$ and $\displaystyle\lim_{x \to a} g(x) = M$, then

4. $\displaystyle\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$.

5. $\displaystyle\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$.

6. $\displaystyle\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LM$.

7. $\displaystyle\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n = L^n$, where $n \geq 2$ is an integer.

8. $\displaystyle\lim_{x \to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$, where $n \geq 2$ is an integer and all roots represent real numbers.

9. $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}, M \neq 0$.

③ Find one-sided limits using properties of limits.

Properties of Limits and Piecewise Functions

In Section 11.1, we used graphs of piecewise functions to find one-sided limits. We can now find such limits by applying properties of limits to the appropriate part of a piecewise function's equation.

EXAMPLE 12 Using Limit Properties to Find One-Sided Limits

Consider the piecewise function defined by

$$f(x) = \begin{cases} x^2 + 5 & \text{if} \quad x < 2 \\ 3x + 1 & \text{if} \quad x \geq 2. \end{cases}$$

Find each of the following limits, or state that the limit does not exist:

a. $\lim\limits_{x \to 2^-} f(x)$ **b.** $\lim\limits_{x \to 2^+} f(x)$ **c.** $\lim\limits_{x \to 2} f(x)$.

Solution

a. To find $\lim\limits_{x \to 2^-} f(x)$, we look at values of $f(x)$ when x is close to 2, but less than 2. Because x is less than 2, we use the first line of the piecewise function's equation, $f(x) = x^2 + 5$.

$$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 + 5) = 2^2 + 5 = 9$$

b. To find $\lim\limits_{x \to 2^+} f(x)$, we look at values of $f(x)$ when x is close to 2, but greater than 2. Because x is greater than 2, we use the second line of the piecewise function's equation, $f(x) = 3x + 1$.

$$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3x + 1) = 3 \cdot 2 + 1 = 7$$

c. We found that

$$\lim\limits_{x \to 2^-} f(x) = 9 \quad \text{and} \quad \lim\limits_{x \to 2^+} f(x) = 7.$$

> The limit as x approaches 2 from the left equals 9.

> The limit as x approaches 2 from the right equals 7.

These one-sided limits are illustrated in **Figure 11.10**. Because the left- and right-hand limits are unequal, $\lim\limits_{x \to 2} f(x)$ does not exist. ●
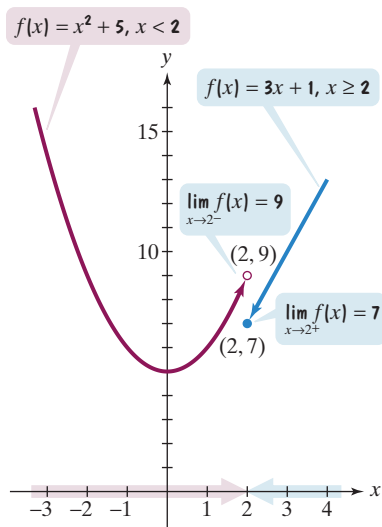
$f(x) = x^2 + 5, \ x < 2$

$f(x) = 3x + 1, \ x \geq 2$

$\lim\limits_{x \to 2^-} f(x) = 9$

$(2, 9)$

$\lim\limits_{x \to 2^+} f(x) = 7$

$(2, 7)$

Figure 11.10

✓ **Check Point 12** Consider the piecewise function defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 1 \\ \sqrt[3]{2x - 1} & \text{if} \quad x \geq 1. \end{cases}$$

Find each of the following limits, or state that the limit does not exist:

a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$.

④ Find limits of fractional expressions in which the limit of the denominator is zero.

Strategies for Finding Limits When the Limit of the Denominator is Zero

When taking the limit of a fractional expression in which the limit of the denominator is zero, the quotient property for limits cannot be used. In such cases, it is necessary to rewrite the expression before the limit can be found. Factoring is one technique that can be used to rewrite an expression.

EXAMPLE 13 Using Factoring to Find a Limit

Find: $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$.

Solution

The limit of the denominator is zero:

$$\lim\limits_{x \to 3} (x - 3) = 3 - 3 = 0.$$

Study Tip

Avoid this common error:

INCORRECT!

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$$
$$= \frac{\lim_{x \to 3}(x^2 - x - 6)}{\lim_{x \to 3}(x - 3)}$$
$$= \frac{3^2 - 3 - 6}{3 - 3} = \frac{0}{0} = 1.$$

Do not use the quotient property if the limit of the denominator is zero. Furthermore, division by zero is not permitted: $\frac{0}{0}$ is *not* equal to 1.
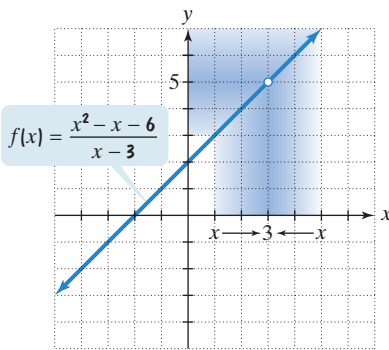

$$f(x) = \frac{x^2 - x - 6}{x - 3}$$

Figure 11.11 As x approaches 3, values of f get closer to 5:

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = 5.$$

Thus, the quotient property for limits cannot be used. Instead, try simplifying the expression using factoring:

$$\frac{x^2 - x - 6}{x - 3} = \frac{(x - 3)(x + 2)}{x - 3}.$$

We seek the limit of this expression as x approaches 3. Because x is close to 3, but not equal to 3, the common factor in the numerator and denominator, $x - 3$, is not equal to zero. With $x - 3 \neq 0$, we can divide the numerator and denominator by $x - 3$. Cancel the common factor, $x - 3$, and then take the limit.

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \to 3}(x + 2) = 3 + 2 = 5 \quad \bullet$$

The graph of $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ is shown in **Figure 11.11**. The hole in the graph at $x = 3$ shows that $f(3)$ is undefined. However, as x approaches 3, the graph shows that the values of f get closer to 5. This verifies the limit that we found in Example 13:

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = 5.$$

✓ **Check Point 13** Find: $\displaystyle\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$.

Rationalizing the numerator or denominator of a fractional expression is another technique that can be used to find a limit when the limit of the denominator is zero.

EXAMPLE 14 **Rationalizing a Numerator to Find a Limit**

Find: $\displaystyle\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$.

Solution As x approaches 0, the denominator of the expression approaches zero. Thus, the quotient property for limits cannot be used. Instead, try rewriting the expression by rationalizing the numerator. If we multiply the numerator and denominator by $\sqrt{4 + x} + 2$, the numerator will not contain radicals.

$$\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$$

$$= \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} \cdot \frac{\sqrt{4 + x} + 2}{\sqrt{4 + x} + 2} \qquad \text{Rationalize the numerator.}$$

$$= \lim_{x \to 0} \frac{(\sqrt{4 + x})^2 - 2^2}{x(\sqrt{4 + x} + 2)} \qquad (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2$$

$$= \lim_{x \to 0} \frac{4 + x - 4}{x(\sqrt{4 + x} + 2)} \qquad (\sqrt{4 + x})^2 = 4 + x$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{4 + x} + 2)} \qquad \text{Simplify: } 4 + x - 4 = x.$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{4 + x} + 2} \qquad \text{Divide both the numerator and denominator by x. This is permitted because x approaches 0, but is not equal to 0.}$$

$$= \frac{\lim_{x \to 0} 1}{\sqrt{\lim_{x \to 0}(4 + x)} + \lim_{x \to 0} 2} \qquad \text{Use limit properties.}$$

$$= \frac{1}{\sqrt{4 + 0} + 2} \qquad \text{Take the limits.}$$

$$= \frac{1}{2 + 2} = \frac{1}{4} \qquad \text{Simplify.} \quad \bullet$$

✓ **Check Point 14** Find: $\lim\limits_{x \to 0} \dfrac{\sqrt{9 + x} - 3}{x}$.

Exercise Set 11.2

Practice Exercises

In Exercises 1–42, use properties of limits to find the indicated limit. It may be necessary to rewrite an expression before limit properties can be applied.

1. $\lim\limits_{x \to 2} 8$

2. $\lim\limits_{x \to 3} (-6)$

3. $\lim\limits_{x \to 2} x$

4. $\lim\limits_{x \to 3} x$

5. $\lim\limits_{x \to 6} (3x - 4)$

6. $\lim\limits_{x \to 7} (4x - 3)$

7. $\lim\limits_{x \to -2} 7x^2$

8. $\lim\limits_{x \to -3} 5x^2$

9. $\lim\limits_{x \to 5} (x^2 - 3x - 4)$

10. $\lim\limits_{x \to 6} (x^2 - 4x - 7)$

11. $\lim\limits_{x \to 2} (5x - 8)^3$

12. $\lim\limits_{x \to 4} (6x - 21)^3$

13. $\lim\limits_{x \to 1} (2x^2 - 3x + 5)^2$

14. $\lim\limits_{x \to 2} (2x^2 + 3x - 1)^2$

15. $\lim\limits_{x \to -4} \sqrt{x^2 + 9}$

16. $\lim\limits_{x \to -1} \sqrt{5x^2 + 4}$

17. $\lim\limits_{x \to 5} \dfrac{x}{x + 1}$

18. $\lim\limits_{x \to 2} \dfrac{3x}{x - 4}$

19. $\lim\limits_{x \to 2} \dfrac{x^2 - 1}{x - 1}$

20. $\lim\limits_{x \to 3} \dfrac{x^2 - 4}{x - 2}$

21. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

22. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

23. $\lim\limits_{x \to 2} \dfrac{2x - 4}{x - 2}$

24. $\lim\limits_{x \to 3} \dfrac{4x - 12}{x - 3}$

25. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 3}{x^2 - 1}$

26. $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 9}$

27. $\lim\limits_{x \to 2} \dfrac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2}$

28. $\lim\limits_{x \to -1} \dfrac{x^3 + 2x^2 + x}{x^4 + x^3 + 2x + 2}$

29. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{x}$

30. $\lim\limits_{x \to 0} \dfrac{\sqrt{16 + x} - 4}{x}$

31. $\lim\limits_{x \to 2} \left[(x + 1)^2 (3x - 1)^3 \right]$

32. $\lim\limits_{x \to -1} \left[(x + 2)^3 (3x + 2) \right]$

33. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

34. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

35. $\lim\limits_{x \to 2} \dfrac{\dfrac{1}{x} - \dfrac{1}{2}}{x - 2}$

36. $\lim\limits_{x \to 3} \dfrac{\dfrac{1}{x} - \dfrac{1}{3}}{x - 3}$

37. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} + 5}{x - 5}$

38. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} + 10}{x - 10}$

39. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x + 3} - \dfrac{1}{3}}{x}$

40. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x}$

41. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8}$

42. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^3 - 1}$

In Exercises 43–50, a piecewise function is given. Use properties of limits to find the indicated limit, or state that the limit does not exist.

43. $f(x) = \begin{cases} x + 5 & \text{if } x < 1 \\ x + 7 & \text{if } x \geq 1 \end{cases}$

    a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

44. $f(x) = \begin{cases} x + 6 & \text{if } x < 1 \\ x + 9 & \text{if } x \geq 1 \end{cases}$

    a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

45. $f(x) = \begin{cases} x^2 + 5 & \text{if } x < 2 \\ x^3 + 1 & \text{if } x \geq 2 \end{cases}$

    a. $\lim\limits_{x \to 2^-} f(x)$ **b.** $\lim\limits_{x \to 2^+} f(x)$ **c.** $\lim\limits_{x \to 2} f(x)$

46. $f(x) = \begin{cases} x^2 + 6 & \text{if } x < 2 \\ x^3 + 2 & \text{if } x \geq 2 \end{cases}$

    a. $\lim\limits_{x \to 2^-} f(x)$ **b.** $\lim\limits_{x \to 2^+} f(x)$ **c.** $\lim\limits_{x \to 2} f(x)$

47. $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 5 & \text{if } x = 3 \end{cases}$

    a. $\lim\limits_{x \to 3^-} f(x)$ **b.** $\lim\limits_{x \to 3^+} f(x)$ **c.** $\lim\limits_{x \to 3} f(x)$

48. $f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & \text{if } x \neq 4 \\ 7 & \text{if } x = 4 \end{cases}$

    a. $\lim\limits_{x \to 4^-} f(x)$ **b.** $\lim\limits_{x \to 4^+} f(x)$ **c.** $\lim\limits_{x \to 4} f(x)$

49. $f(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ x^2 - 1 & \text{if } x > 1 \end{cases}$

    a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

50. $f(x) = \begin{cases} 4 - x & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ x^2 + 2 & \text{if } x > 1 \end{cases}$

    a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

Practice Plus

51. Let $f(x) = x^3 - x^2 + 5x - 1$ and $g(x) = 2$. Find $\lim\limits_{x \to 3} (f \circ g)(x)$ and $\lim\limits_{x \to 3} (g \circ f)(x)$.

52. Let $f(x) = x^3 + x^2 - 6x - 1$ and $g(x) = 3$. Find $\lim\limits_{x \to 4} (f \circ g)(x)$ and $\lim\limits_{x \to 4} (g \circ f)(x)$.

53. Let $f(x) = \dfrac{2}{x}$ and $g(x) = \dfrac{3}{x-1}$.

Find $\lim\limits_{x\to 1}(f \circ g)(x)$ and $\lim\limits_{x\to 1}(g \circ f)(x)$.

54. Let $f(x) = \dfrac{4}{x-1}$ and $g(x) = \dfrac{1}{x+2}$.

Find $\lim\limits_{x\to 1}(f \circ g)(x)$ and $\lim\limits_{x\to 1}(g \circ f)(x)$.

**55.** Let $f(x) = x^2 + 4$, $x \geq 0$. Find $\lim\limits_{x\to 8} f^{-1}(x)$.

56. Let $f(x) = x^2 + 9$, $x \geq 0$. Find $\lim\limits_{x\to 25} f^{-1}(x)$.

**57.** Let $f(x) = \dfrac{2x+1}{x-1}$. Find $\lim\limits_{x\to 4} f^{-1}(x)$.

58. Let $f(x) = \dfrac{2x+3}{x+4}$. Find $\lim\limits_{x\to 5} f^{-1}(x)$.

Application Exercises

In Albert Einstein's special theory of relativity, time slows down and length in the direction of motion decreases from the point of view of an observer watching an object moving at velocities approaching the speed of light. (The speed of light is approximately 186,000 miles per second. At this speed, a beam of light can travel around the world about seven times in a single second.) Einstein's theory, verified with experiments in atomic physics, forms the basis of Exercises 59–60.

600 meters

Starship at rest

84 meters
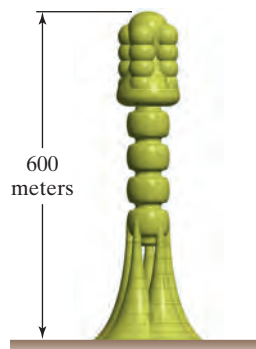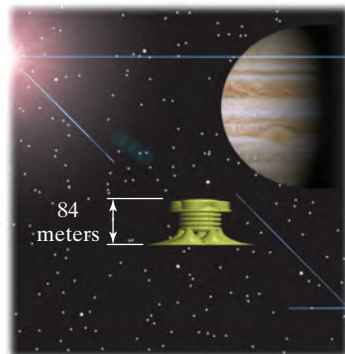
Change in starship's length
when moving at 99% of light's speed
when viewed by an observer

59. The formula

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

expresses the length, L, of a starship moving at velocity v with respect to an observer on Earth, where L_0 is the length of the starship at rest and c is the speed of light.

a. Find $\lim\limits_{v\to c^-} L$.

b. If a starship is traveling at velocities approaching the speed of light, what does the limit in part (a) indicate about its length from the perspective of a stationary viewer on Earth?

c. Explain why a left-hand limit is used in part (a).

60. The formula

$$R_a = R_f\sqrt{1 - \frac{v^2}{c^2}}$$

expresses the aging rate of an astronaut, R_a, relative to the aging rate of a friend on Earth, R_f, where v is the astronaut's velocity and c is the speed of light.

a. Find $\lim\limits_{v\to c^-} R_a$.

b. If you are traveling in a starship at velocities approaching the speed of light, what does the limit in part (a) indicate about your aging rate relative to a friend on Earth?

c. Explain why a left-hand limit is used in part (a).

Writing in Mathematics

61. Explain how to find the limit of a constant. Then express your written explanation using limit notation.

62. Explain how to find the limit of the identity function $f(x) = x$. Then express your written explanation using limit notation.

63. Explain how to find the limit of a sum. Then express your written explanation using limit notation.

64. Explain how to find the limit of a difference. Then express your written explanation using limit notation.

65. Explain how to find the limit of a product. Then express your written explanation using limit notation.

66. Describe how to find the limit of a polynomial function. Provide an example with your description.

67. Explain how to find the following limit: $\lim\limits_{x\to 2}(3x^2 - 10)^3$. Then use limit notation to write the limit property that supports your explanation.

68. Explain how to find the following limit: $\lim\limits_{x\to 2}\sqrt{5x - 6}$. Then use limit notation to write the limit property that supports your explanation.

69. Explain how to find the limit of a quotient if the limit of the denominator is not zero. Then express your written explanation using limit notation.

70. Write an example involving the limit of a quotient in which the quotient property for limits cannot be applied. Explain why the property cannot be applied to your limit problem.

71. Explain why

$$\lim\limits_{x\to 4} \frac{(x+4)(x-4)}{x-4}$$

can be found by first dividing the numerator and the denominator of the expression by $x - 4$. Division by zero is undefined. How can we be sure that we are not dividing the numerator and the denominator by zero?

Technology Exercises

72. Use the $\boxed{\text{TABLE}}$ feature of your graphing utility to verify any five of the limits that you found in Exercises 1–42.

73. Use the $\boxed{\text{ZOOM IN}}$ feature of your graphing utility to verify any five of the limits that you found in Exercises 1–42. Zoom in on the graph of the given function, f, near $x = a$ to verify each limit.

Critical Thinking Exercises

Make Sense? *In Exercises 74–77, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

74. I evaluated a polynomial function f at 3 and obtained 7, so 7 must be $\lim\limits_{x \to 3} f(x)$.

75. I'm working with functions f and g for which $\lim\limits_{x \to 4} f(x) = 0$, $\lim\limits_{x \to 4} g(x) = -5$, and $\lim\limits_{x \to 4} [f(x) - g(x)] = 5$.

76. I'm working with functions f and g for which $\lim\limits_{x \to 4} f(x) = 0$, $\lim\limits_{x \to 4} g(x) = -5$, and $\lim\limits_{x \to 4} \dfrac{f(x)}{g(x)} = 0$.

77. I'm working with functions f and g for which $\lim\limits_{x \to 4} f(x) = 0$, $\lim\limits_{x \to 4} g(x) = -5$, and $\lim\limits_{x \to 4} \dfrac{g(x)}{f(x)} \neq 0$.

In Exercises 78–79, find the indicated limit.

78. $\lim\limits_{x \to 0} x\left(1 - \dfrac{1}{x}\right)$ **79.** $\lim\limits_{x \to 4} \left(\dfrac{1}{x} - \dfrac{1}{4}\right)\left(\dfrac{1}{x - 4}\right)$

In Exercises 80–81, find $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$.

80. $f(x) = x^2 + 2x - 3, a = 1$ **81.** $f(x) = \sqrt{x}, a = 1$

In Exercises 82–83, use properties of limits and the following limits

$$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1, \qquad \lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0,$$
$$\lim\limits_{x \to 0} \sin x = 0, \qquad \lim\limits_{x \to 0} \cos x = 1$$

to find the indicated limit.

82. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$ **83.** $\lim\limits_{x \to 0} \dfrac{2 \sin x + \cos x - 1}{3x}$

Group Exercises

84. In the next column is a list of ten common errors involving algebra, trigonometry, and limits that students frequently make in calculus. Group members should examine each error and describe the mistake. Where possible, correct each error.

Finally, group members should offer suggestions for avoiding each error.

a. $(x + h)^3 - x^3 = x^3 + h^3 - x^3 = h^3$

b. $\dfrac{1}{a + b} = \dfrac{1}{a} + \dfrac{1}{b}$

c. $\dfrac{1}{a + b} = \dfrac{1}{a} + b$

d. $\sqrt{x + h} - \sqrt{x} = \sqrt{x} + \sqrt{h} - \sqrt{x} = \sqrt{h}$

e. $\dfrac{\sin 2x}{x} = \sin 2$

f. $\dfrac{a + bx}{a} = 1 + bx$

g. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1} = \dfrac{1^3 - 1}{1 - 1} = \dfrac{0}{0} = 1$

h. $\sin(x + h) - \sin x = \sin x + \sin h - \sin x = \sin h$

i. $ax = bx$, so $a = b$

j. To find $\lim\limits_{x \to 4} \dfrac{x^2 - 9}{x - 3}$, it is necessary to rewrite $\dfrac{x^2 - 9}{x - 3}$ by factoring $x^2 - 9$.

85. Research and present a group report about the history of the feud between Newton and Leibniz over who invented calculus. What other interests did these men have in addition to mathematics? What practical problems led them to the invention of calculus? What were their personalities like? Whose version established the notation and rules of calculus that we use today?

Preview Exercises

Exercises 86–88 will help you prepare for the material covered in the next section. In each exercise, use what occurs near 3 and at 3 to graph the function in an open interval about 3. (Graphs will vary.) Is it necessary to lift your pencil off the paper to obtain each graph? Explain your answer.

86. $\lim\limits_{x \to 3} f(x) = 5; f(3) = 5$ **87.** $\lim\limits_{x \to 3} f(x) = 5; f(3) = 6$

88. $\lim\limits_{x \to 3^-} f(x) = 5; \lim\limits_{x \to 3^+} f(x) = 6; f(3) = 5$

Section 11.3 Limits and Continuity

Objectives

1 Determine whether a function is continuous at a number.

2 Determine for what numbers a function is discontinuous.

Why you should not ski down discontinuous slopes

In everyday speech, a continuous process is one that goes on without interruption and without abrupt changes. In mathematics, a continuous function has much the same meaning. The graph of a continuous function does not have interrupting breaks, such as holes, gaps, or jumps. Thus, the graph of a continuous function can be drawn without lifting a pencil off the paper. In this section, you will learn how limits can be used to describe continuity.