91. The graph of $(x - 3)^2 + (y + 5)^2 = 36$ is a circle with radius 6 centered at $(-3, 5)$.

92. The graph of $(x - 4) + (y + 6) = 25$ is a circle with radius 5 centered at $(4, -6)$.

93. The graph of $(x - 3)^2 + (y + 5)^2 = -36$ is a circle with radius 6 centered at $(3, -5)$.

94. Show that the points $A(1, 1 + d)$, $B(3, 3 + d)$, and $C(6, 6 + d)$ are collinear (lie along a straight line) by showing that the distance from A to B plus the distance from B to C equals the distance from A to C.

95. Prove the midpoint formula by using the following procedure.

a. Show that the distance between (x_1, y_1) and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is equal to the distance between (x_2, y_2) and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

b. Use the procedure from Exercise 94 and the distances from part (a) to show that the points (x_1, y_1), $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, and (x_2, y_2) are collinear.

96. Find the area of the donut-shaped region bounded by the graphs of $(x - 2)^2 + (y + 3)^2 = 25$ and $(x - 2)^2 + (y + 3)^2 = 36$.

97. A **tangent line** to a circle is a line that intersects the circle at exactly one point. The tangent line is perpendicular to the radius of the circle at this point of contact. Write an equation in point-slope form for the line tangent to the circle whose equation is $x^2 + y^2 = 25$ at the point $(3, -4)$.

Preview Exercises

Exercises 98–100 will help you prepare for the material covered in the next section.

98. Write an algebraic expression for the fare increase if a \$200 plane ticket is increased to x dollars.

99. Find the perimeter and the area of each rectangle with the given dimensions:

a. 40 yards by 30 yards **b.** 50 yards by 20 yards.

100. Solve for h: $\pi r^2 h = 22$. Then rewrite $2\pi r^2 + 2\pi rh$ in terms of r.

Section 1.10 Modeling with Functions

Objectives

1. Construct functions from verbal descriptions.
2. Construct functions from formulas.

A can of Coca-Cola is sold every six seconds throughout the world.

In 2005, to curb consumption of sugared soda, the Center for Science in the Public Interest (CSPI) urged the FDA to slap cigarette-style warning labels on these drinks, calling them "liquid candy." Despite the variety of its reputations throughout the world, the soft drink industry has spent far more time reducing the amount of aluminum in its cylindrical cans than addressing the problems of the nutritional disaster floating within its packaging. In the 1960s, one pound of aluminum made fewer than 20 cans; today, almost 30 cans come out of the same amount. The thickness of the can wall is less than five-thousandths of an inch, about the same as a magazine cover.

Many real-world problems involve constructing mathematical models that are functions. The problem of minimizing the amount of aluminum needed to manufacture a 12-ounce soft-drink can first requires that we express the surface area of all such cans as a function of their radius. In constructing such a function, we must be able to translate a verbal description into a mathematical representation—that is, a mathematical model.

Study Tip

In calculus, you will solve problems involving maximum or minimum values of functions. Such problems often require creating the function that is to be maximized or minimized. Quite often, the calculus is fairly routine. It is the algebraic setting up of the function that causes difficulty. That is why the material in this section is so important.

In Chapter P, Section P.8, we reviewed how to obtain equations that modeled problems' verbal conditions. Earlier in this chapter, we used real-world data to obtain models that were linear functions. In this section, you will learn to use verbal descriptions and formulas to obtain models involving both linear and nonlinear functions.

1 Construct functions from verbal descriptions.

Functions from Verbal Descriptions

There is no rigid step-by-step procedure that can be used to construct a function from a verbal description. Read the problem carefully. Attempt to write a critical sentence that describes the function's conditions in terms of its independent variable, x. In the following examples, we will use voice balloons that show these critical sentences, or **verbal models**. Then translate the verbal model into the algebraic notation used to represent a function's equation.

EXAMPLE 1 Modeling Costs of Long-Distance Carriers

You are choosing between two long-distance telephone plans. Plan A has a monthly fee of \$20 with a charge of \$0.05 per minute for all long-distance calls. Plan B has a monthly fee of \$5 with a charge of \$0.10 per minute for all long-distance calls.

a. Express the monthly cost for plan A, f, as a function of the number of minutes of long-distance calls in a month, x.

b. Express the monthly cost for plan B, g, as a function of the number of minutes of long-distance calls in a month, x.

c. For how many minutes of long-distance calls will the costs for the two plans be the same?

Solution

a. The monthly cost for plan A is the monthly fee, \$20, plus the per-minute charge, \$0.05, times the number of minutes of long-distance calls, x.

Monthly cost for plan A	equals	monthly fee	plus	per-minute charge	times	the number of minutes of long-distance calls.
$f(x)$	$=$	20	$+$	0.05	$\cdot$	x

The function $f(x) = 0.05x + 20$, expressed in slope-intercept form, models the monthly cost, in dollars, in terms of the number of minutes of long-distance calls, x.

b. The monthly cost for plan B is the monthly fee, \$5, plus the per-minute charge, \$0.10, times the number of minutes of long-distance calls, x.

Monthly cost for plan B	equals	monthly fee	plus	per-minute charge	times	the number of minutes of long-distance calls.
$g(x)$	$=$	5	$+$	0.10	$\cdot$	x

The function $g(x) = 0.10x + 5$, expressed in slope-intercept form, models the monthly cost, g, in dollars, in terms of the number of minutes of long-distance calls, x.

c. We are interested in how many minutes of long-distance calls, x, result in the same monthly costs, f and g, for the two plans. Thus, we must set the equations for f and g equal to each other. We then solve the resulting linear equation for x.

The monthly cost for plan A | must equal | the monthly cost for plan B.

$$20 + 0.05x \quad = \quad 5 + 0.10x$$

$$20 + 0.05x = 5 + 0.10x \quad \text{This equation models equal monthly costs.}$$

$$20 = 5 + 0.05x \quad \text{Subtract 0.05x from both sides.}$$

$$15 = 0.05x \quad \text{Subtract 5 from both sides.}$$

$$300 = x \quad \text{Divide both sides by 0.05: } \frac{15}{0.05} = 300.$$

The costs for the two plans will be the same with 300 minutes of long-distance calls. Take a moment to verify that $f(300) = g(300) = 35$. Thus, the cost for each plan will be \$35.

In Example 1, the functions modeling the costs for the two plans are both linear functions of the form $f(x) = mx + b$. Based on the meaning of the functions' variables, we can interpret slope and y-intercept as follows:

Plan A

$$f(x) = 0.05x + 20$$

The slope indicates that the rate of change in the plan's cost is \$0.05 per minute.

The y-intercept indicates the starting cost with no long-distance calls is \$20.

Plan B

$$g(x) = 0.10x + 5.$$

The slope indicates that the rate of change in the plan's cost is \$0.10 per minute.

The y-intercept indicates the starting cost with no long-distance calls is \$5.

Technology

Numeric and Graphic Connections

We can use a graphing utility to numerically or graphically verify our work in Example 1(c). Enter the linear functions that model the costs for the two plans.

The monthly cost for plan A | must equal | the monthly cost for plan B.

$$20 + 0.05x \quad = \quad 5 + 0.10x$$

Enter $y_1 = 20 + .05x$. | Enter $y_2 = 5 + .10x$.

Numeric Check

Display a table for y_1 and y_2.

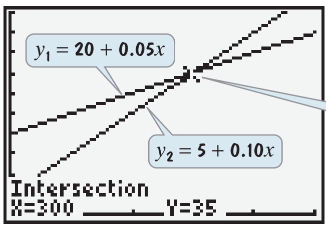
X	Y1	Y2
100	25	15
150	27.5	20
200	30	25
250	32.5	30
300	35	35
350	37.5	40
400	40	45

X=300

When $x = 300$, y_1 and y_2 have the same value, 35. With 300 minutes of calls, costs are the same, \$35, for both plans.

Graphic Check

Display graphs for y_1 and y_2. Use the intersection feature.



Graphs intersect at (300, 35). With 300 minutes of calls, costs are the same, \$35, for both plans.

[0, 500, 100] by [0, 50, 5]

Check Point 1 You are choosing between two long-distance telephone plans. Plan A has a monthly fee of \$15 with a charge of \$0.08 per minute for all long-distance calls. Plan B has a monthly fee of \$3 with a charge of \$0.12 per minute for all long-distance calls.

a. Express the monthly cost for plan A, f, as a function of the number of minutes of long-distance calls in a month, x.

b. Express the monthly cost for plan B, g, as a function of the number of minutes of long-distance calls in a month, x.

c. For how many minutes of long-distance calls will the costs for the two plans be the same?

EXAMPLE 2 Modeling the Number of Customers and Revenue

On a certain route, an airline carries 6000 passengers per month, each paying \$200. A market survey indicates that for each \$1 increase in the ticket price, the airline will lose 100 passengers.

a. Express the number of passengers per month, N, as a function of the ticket price, x.

b. The airline's monthly revenue for the route is the product of the number of passengers and the ticket price. Express the monthly revenue, R, as a function of the ticket price, x.

Solution

a. The number of passengers, N, depends on the ticket price, x. In particular, the number of passengers is the original number, 6000, minus the number lost to the fare increase. The following table shows how to find the number lost to the fare increase:

English Phrase	Algebraic Translation
Ticket price	x
Amount of fare increase: ticket price minus original ticket price	$x - 200$ (The original ticket price was \$200.)
Decrease in passengers due to the fare increase: 100 times the dollar amount of the fare increase	$100(x - 200)$ (100 passengers are lost for each dollar of fare increase.)

The number of passengers per month, N, is the original number of passengers, 6000, minus the decrease due to the fare increase.

Number of passengers per month | equals | the original number of passengers | minus | the decrease in passengers due to the fare increase.

$$\begin{aligned} N(x) &= 6000 - 100(x - 200) \\ &= 6000 - 100x + 20{,}000 \\ &= -100x + 26{,}000 \end{aligned}$$

The linear function $N(x) = -100x + 26{,}000$ models the number of passengers per month, N, in terms of the price per ticket, x. The linear function's slope, -100, indicates that the rate of change is a loss of 100 passengers per dollar of fare increase.

b. The monthly revenue for the route is the number of passengers, $-100x + 26{,}000$, times the ticket price, x.

Monthly revenue | equals | the number of passengers | times | the ticket price.

$$R(x) = (-100x + 26{,}000) \cdot x$$
$$= -100x^2 + 26{,}000x$$

The function $R(x) = -100x^2 + 26{,}000x$ models the airline's monthly revenue for the route, R, in terms of the ticket price, x.

The revenue function in Example 2 is of the form $f(x) = ax^2 + bx + c$. Any function of this form, where $a \neq 0$, is called a **quadratic function**. In this chapter, we used the bowl-shaped graph of the standard quadratic function, $f(x) = x^2$, to graph various transformations. In the next chapter, you will study quadratic functions in detail, including where maximum or minimum values occur.

Check Point 2 On a certain route, an airline carries 8000 passengers per month, each paying \$100. A market survey indicates that for each \$1 increase in ticket price, the airline will lose 100 passengers.

a. Express the number of passengers per month, N, as a function of the ticket price, x.

b. Express the monthly revenue for the route, R, as a function of the ticket price, x.

2 Construct functions from formulas.

Study Tip

When developing functions that model geometric situations, it is helpful to draw a diagram. Label the given and variable quantities on the diagram.

Functions from Formulas

In Chapter P, Section P.8, we used basic geometric formulas to obtain equations that modeled geometric situations. Formulas for area, perimeter, and volume are given in Table P.6 on page 106. Obtaining functions that model geometric situations requires a knowledge of these formulas. Take a moment to turn to page 106 and be sure that you are familiar with the 13 formulas given in the table.

In our next example, we will obtain a function using the formula for the volume of a rectangular solid, $V = lwh$. A rectangular solid's volume is the product of its length, width, and height.

EXAMPLE 3 Obtaining a Function from a Geometric Formula

A machine produces open boxes using square sheets of metal measuring 12 inches on each side. The machine cuts equal-sized squares from each corner. Then it shapes the metal into an open box by turning up the sides.

a. Express the volume of the box, V, in cubic inches, as a function of the length of the side of the square cut from each corner, x, in inches.
b. Find the domain of V.

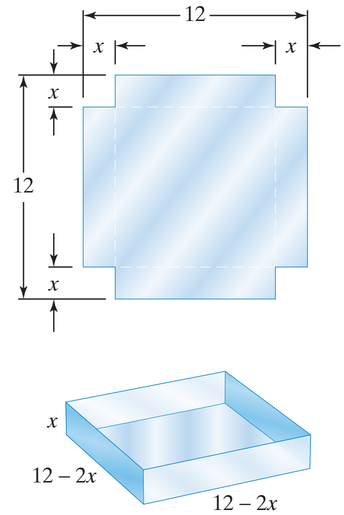


Figure 1.80
Producing open boxes using square sheets of metal

Solution

a. The situation is illustrated in **Figure 1.80**. The volume of the box in the lower portion of the figure is the product of its length, width, and height. The height of the box is the same as the side of the square cut from each corner, x. Because the 12-inch square has x inches cut from each corner, the length of the resulting box is $12 - x - x$, or $12 - 2x$. Similarly, the width of the resulting box is also $12 - 2x$.

Volume of the box | equals | length of the box | times | width of the box | times | height of the box.

$$V(x) = (12 - 2x) \cdot (12 - 2x) \cdot x$$

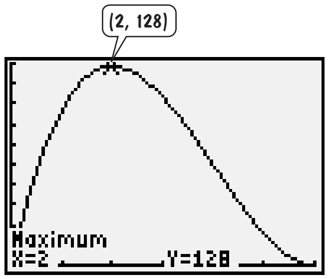
The function $V(x) = x(12 - 2x)^2$ models the volume of the box, V, in terms of the length of the side of the square cut from each corner, x.

b. The formula for V involves a polynomial, $x(12 - 2x)^2$, which is defined for any real number, x. However, in the function $V(x) = x(12 - 2x)^2$, x represents the number of inches cut from each corner of the 12-inch square. Thus, $x > 0$. To produce an open box, the machine must cut less than 6 inches from each corner of the 12-inch square. Thus, $x < 6$. The domain of V is $\{x | 0 < x < 6\}$, or, in interval notation, $(0, 6)$.

Technology

Graphic Connections

The graph of the function $V(x) = x(12 - 2x)^2$, the model for the volume of the box in **Figure 1.80**, is shown in a $[0, 6, 1]$ by $[0, 130, 13]$ viewing rectangle. The graphing utility's maximum function feature indicates that the volume of the box is a maximum, 128 cubic inches, when the side of the square cut from each corner of the metal sheet is 2 inches.



Check Point 3 A machine produces open boxes using rectangular sheets of metal measuring 15 inches by 8 inches. The machine cuts equal-sized squares from each corner. Then it shapes the metal into an open box by turning up the sides.

a. Express the volume of the box, V, in cubic inches, as a function of the length of the side of the square cut from each corner, x, in inches.

b. Find the domain of V.

In many situations, the conditions of the problem result in a function whose equation contains more than one variable. If this occurs, use the given information to write an equation among these variables. Then use this equation to eliminate all but one of the variables in the function's expression.

EXAMPLE 4 Modeling the Area of a Rectangle with a Fixed Perimeter

You have 140 yards of fencing to enclose a rectangular garden. Express the area of the garden, A, as a function of one of its dimensions, x.

Solution Because you have 140 yards of fencing, **Figure 1.81** illustrates three of your options for enclosing the garden. In each case, the perimeter of the rectangle, twice the length plus twice the width, is 140 yards. By contrast, the area, length times width, varies according to the length of a side, x.

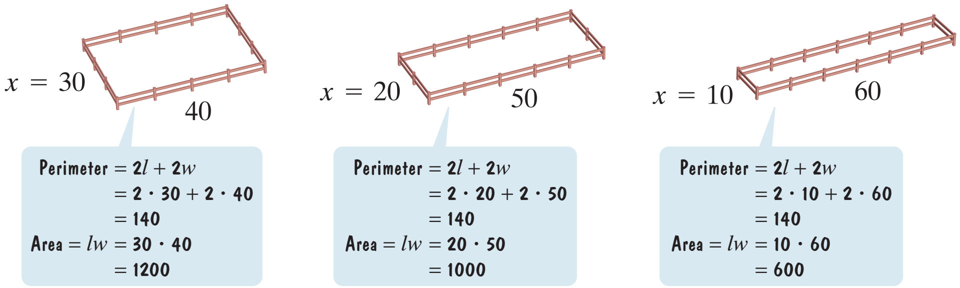


Figure 1.81 Rectangles with a fixed perimeter and varying areas

As specified, x represents one of the dimensions of the rectangle. In particular, let

$$x = \text{the length of the garden}$$
$$y = \text{the width of the garden.}$$

The area, A, of the garden is the product of its length and its width:

$$A = xy.$$

There are two variables in this formula—the garden's length, x, and its width, y. We need to transform this into a function in which A is represented by one variable, x, the garden's length. Thus, we must express the width, y, in terms of the length, x. We do this using the information that you have 140 yards of fencing.

$$2x + 2y = 140$$ The perimeter, twice the length plus twice the width, is 140 yards.

$$2y = 140 - 2x$$ Subtract $2x$ from both sides.

$$y = \frac{140 - 2x}{2}$$ Divide both sides by 2.

$$y = 70 - x$$ Divide each term in the numerator by 2.

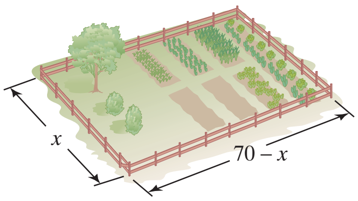


Figure 1.82

Now we substitute $70 - x$ for y in the formula for area.

$$A = xy = x(70 - x)$$

The rectangle and its dimensions are illustrated in **Figure 1.82**. Because A is a function of x, we can write

$$A(x) = x(70 - x) \quad \text{or} \quad A(x) = 70x - x^2.$$

This function models the area, A, of a rectangular garden with a perimeter of 140 yards in terms of the length of a side, x.

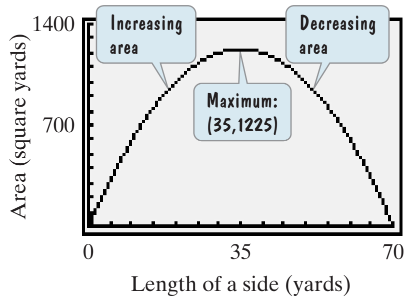
Technology

Graphic Connections

The graph of the function

$$A(x) = x(70 - x),$$

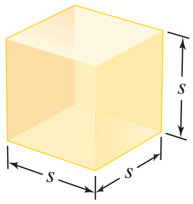
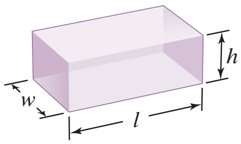
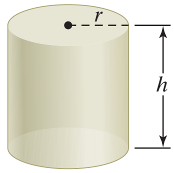
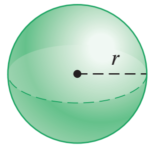
the model for the area of the garden in Example 4, is shown in a [0, 70, 5] by [0, 1400, 100] viewing rectangle. The graph shows that as the length of a side increases, the enclosed area increases, then decreases. The area of the garden is a maximum, 1225 square yards, when the length of one of its sides is 35 yards.



Check Point 4 You have 200 feet of fencing to enclose a rectangular garden. Express the area of the garden, A, as a function of one of its dimensions, x.

In order to save on production costs, manufacturers need to use the least amount of material for containers that are required to hold a specified volume of their product. Using the least amount of material involves minimizing the surface area of the container. Formulas for surface area, A, are given in **Table 1.5.**

Table 1.5 Common Formulas for Surface Area

Cube	Rectangular Solid	Circular Cylinder	Sphere
$A = 6s^2$	$A = 2lw + 2lh + 2wh$	$A = 2\pi r^2 + 2\pi rh$	$A = 4\pi r^2$
			

Technology

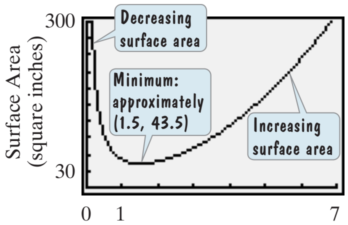
Graphic Connections

The graph of the function

$$A(r) = 2\pi r^2 + \frac{44}{r}, \text{ or}$$

$$y = 2\pi x^2 + \frac{44}{x},$$

the model for the surface area of the soft-drink can in **Figure 1.83**, was obtained with a graphing utility.



[0, 7, 1] by [0, 300, 30]

As the radius increases, the can's surface area decreases, then increases. Using a graphing utility's minimum function feature, it can be shown that when $x \approx 1.5$, the value of y is smallest ($y \approx 43.5$). Thus, the least amount of material needed to manufacture the can, approximately 43.5 square inches, occurs when its radius is approximately 1.5 inches. In calculus, you will learn techniques that give the exact radius needed to minimize the can's surface area.

EXAMPLE 5 Modeling the Surface Area of a Soft-Drink Can with Fixed Volume

Figure 1.83 shows a cylindrical soft-drink can. The can is to have a volume of 12 fluid ounces, approximately 22 cubic inches. Express the surface area of the can, A, in square inches, as a function of its radius, r, in inches.

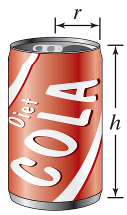


Figure 1.83

Solution The surface area, A, of the cylindrical can in **Figure 1.83** is given by

$$A = 2\pi r^2 + 2\pi rh.$$

There are two variables in this formula—the can's radius, r, and its height, h. We need to transform this into a function in which A is represented by one variable, r, the radius of the can. Thus, we must express the height, h, in terms of the radius, r. We do this using the information that the can's volume, $V = \pi r^2 h$, must be 22 cubic inches.

$$\pi r^2 h = 22 \quad \text{The volume of the can is 22 cubic inches.}$$

$$h = \frac{22}{\pi r^2} \quad \text{Divide both sides by } \pi r^2 \text{ and solve for } h.$$

Now we substitute $\frac{22}{\pi r^2}$ for h in the formula for surface area.

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\left(\frac{22}{\pi r^2}\right) = 2\pi r^2 + \frac{44}{r}$$

Because A is a function of r, the can's radius, we can express the surface area of the can as

$$A(r) = 2\pi r^2 + \frac{44}{r}.$$

Check Point 5 A cylindrical can is to hold 1 liter, or 1000 cubic centimeters, of oil. Express the surface area of the can, A, in square centimeters, as a function of its radius, r, in centimeters.

Our next example involves constructing a function that models simple interest. The annual simple interest that an investment earns is given by the formula

$$I = Pr,$$

where I is the simple interest, P is the principal, and r is the simple interest rate, expressed in decimal form. Suppose, for example, that you deposit \$2000 $(P = 2000)$ in an account that has a simple interest rate of 3% $(r = 0.03)$. The annual simple interest is computed as follows:

$$I = Pr = (2000)(0.03) = 60.$$

The annual interest is \$60.

EXAMPLE 6 Modeling Simple Interest

You inherit \$16,000 with the stipulation that for the first year the money must be placed in two investments expected to pay 6% and 8% annual interest. Express the expected interest, I, as a function of the amount of money invested at 6%, x.

Solution As specified, x represents the amount invested at 6%. We will let y represent the amount invested at 8%. The expected interest, I, on the two investments combined is the expected interest on the 6% investment plus the expected interest on the 8% investment.

Expected interest on the 6% investment | Expected interest on the 8% investment

$$I = 0.06x + 0.08y$$

Rate times principal | Rate times principal

There are two variables in this formula—the amount invested at 6%, x, and the amount invested at 8%, y. We need to transform this into a function in which I is represented by one variable, x, the amount invested at 6%. Thus, we must express the amount invested at 8%, y, in terms of the amount invested at 6%, x. We do this using the information that you have \$16,000 to invest.

$$x + y = 16{,}000 \quad \text{The sum of the amounts invested at each rate must be \$16,000.}$$

$$y = 16{,}000 - x \quad \text{Subtract } x \text{ from both sides and solve for } y.$$

Now we substitute $16{,}000 - x$ for y in the formula for interest.

$$I = 0.06x + 0.08y = 0.06x + 0.08(16{,}000 - x).$$

Because I is now a function of x, the amount invested at 6%, the expected interest can be expressed as

$$I(x) = 0.06x + 0.08(16{,}000 - x).$$

Check Point 6 You place \$25,000 in two investments expected to pay 7% and 9% annual interest. Express the expected interest, I, as a function of the amount of money invested at 7%, x.

Our next example involves constructing a function using the distance formula. In the previous section, we saw that the distance between two points in the rectangular coordinate system is the square root of the difference between their x-coordinates squared plus the difference between their y-coordinates squared.

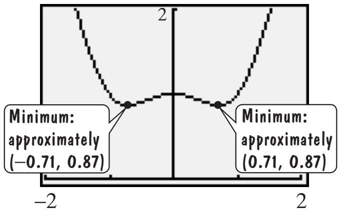
Technology

Graphic Connections

The graph of the function

$$d(x) = \sqrt{x^4 - x^2 + 1}, \text{ or}$$
$$y = \sqrt{x^4 - x^2 + 1},$$

the model for the distance, d, shown in **Figure 1.84**, was obtained with a graphing utility.



[−2, 2, 1] by [0, 2, 1]

Using a graphing utility's minimum function feature, it can be shown that when $x \approx -0.71$ and when $x \approx 0.71$, the value of d is smallest ($d \approx 0.87$ is a relative minimum). Examine **Figure 1.84**. The graph's y-axis symmetry indicates that there are two points on $y = 1 - x^2$ whose distance to the origin is smallest.

EXAMPLE 7 Modeling the Distance from the Origin to a Point on a Graph

Figure 1.84 shows that $P(x, y)$ is a point on the graph of $y = 1 - x^2$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

Solution We use the distance formula to find the distance, d, from $P(x, y)$ to the origin, $(0, 0)$.

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

There are two variables in this formula—the point's x-coordinate and its y-coordinate. We need to transform this into a function in which d is represented by one variable, x, the point's x-coordinate. Thus, we must express y in terms of x. We do this using the information shown in **Figure 1.84**, namely that $P(x, y)$ is a point on the graph of $y = 1 - x^2$. This means that we can replace y with $1 - x^2$ in our formula for d.

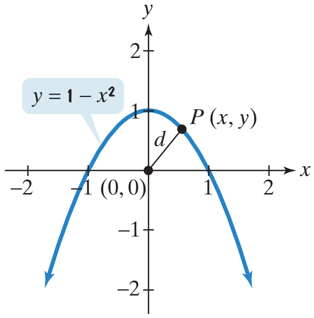


Figure 1.84

$$d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (1 - x^2)^2} = \sqrt{x^2 + 1 - 2x^2 + x^4} = \sqrt{x^4 - x^2 + 1}$$

Square $1 - x^2$ using $(A - B)^2 = A^2 - 2AB + B^2$.

The distance, d, from $P(x, y)$ to the origin can be expressed as a function of the point's x-coordinate as

$$d(x) = \sqrt{x^4 - x^2 + 1}.$$

Check Point 7 Let $P(x, y)$ be a point on the graph of $y = x^3$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

Exercise Set 1.10

Practice and Application Exercises

1. A car rental agency charges \$200 per week plus \$0.15 per mile to rent a car.

a. Express the weekly cost to rent the car, f, as a function of the number of miles driven during the week, x.

b. How many miles did you drive during the week if the weekly cost to rent the car was \$320?

2. A car rental agency charges \$180 per week plus \$0.25 per mile to rent a car.

a. Express the weekly cost to rent the car, f, as a function of the number of miles driven during the week, x.

b. How many miles did you drive during the week if the weekly cost to rent the car was \$395?

3. One yardstick for measuring how steadily—if slowly—athletic performance has improved is the mile run. In 1954, Roger Bannister of Britain cracked the 4-minute mark, setting the record for running a mile in 3 minutes, 59.4 seconds, or 239.4 seconds. In the half-century since then, the record has decreased by 0.3 second per year.

a. Express the record time for the mile run, M, as a function of the number of years after 1954, x.

b. If this trend continues, in which year will someone run a 3-minute, or 180-second, mile?

4. According to the National Center for Health Statistics, in 1990, 28% of babies in the United States were born to parents who were not married. Throughout the 1990s, this percentage increased by approximately 0.6 per year.

a. Express the percentage of babies born out of wedlock, P, as a function of the number of years after 1990, x.

b. If this trend continues, in which year will 40% of babies be born out of wedlock?

5. The bus fare in a city is \$1.25. People who use the bus have the option of purchasing a monthly discount pass for \$21.00. With the discount pass, the fare is reduced to \$0.50.

a. Express the total monthly cost to use the bus without a discount pass, f, as a function of the number of times in a month the bus is used, x.

b. Express the total monthly cost to use the bus with a discount pass, g, as a function of the number of times in a month the bus is used, x.

c. Determine the number of times in a month the bus must be used so that the total monthly cost without the discount pass is the same as the total monthly cost with the discount pass. What will be the monthly cost for each option?

6. A discount pass for a bridge costs \$21 per month. The toll for the bridge is normally \$2.50, but it is reduced to \$1 for people who have purchased the discount pass.
   a. Express the total monthly cost to use the bridge without a discount pass, f, as a function of the number of times in a month the bridge is crossed, x.
   b. Express the total monthly cost to use the bridge with a discount pass, g, as a function of the number of times in a month the bridge is crossed, x.
   c. Determine the number of times in a month the bridge must be crossed so that the total monthly cost without the discount pass is the same as the total monthly cost with the discount pass. What will be the monthly cost for each option?

7. You are choosing between two plans at a discount warehouse. Plan A offers an annual membership of \$100 and you pay 80% of the manufacturer's recommended list price. Plan B offers an annual membership fee of \$40 and you pay 90% of the manufacturer's recommended list price.
   a. Express the total yearly amount paid to the warehouse under plan A, f, as a function of the dollars of merchandise purchased during the year, x.
   b. Express the total yearly amount paid to the warehouse under plan B, g, as a function of the dollars of merchandise purchased during the year, x.
   c. How many dollars of merchandise would you have to purchase in a year to pay the same amount under both plans? What will be the total yearly amount paid to the warehouse for each plan?

8. You are choosing between two plans at a discount warehouse. Plan A offers an annual membership fee of \$300 and you pay 70% of the manufacturer's recommended list price. Plan B offers an annual membership fee of \$40 and you pay 90% of the manufacturer's recommended list price.
   a. Express the total yearly amount paid to the warehouse under plan A, f, as a function of the dollars of merchandise purchased during the year, x.
   b. Express the total yearly amount paid to the warehouse under plan B, g, as a function of the dollars of merchandise purchased during the year, x.
   c. How many dollars of merchandise would you have to purchase in a year to pay the same amount under both plans? What will be the total yearly amount paid to the warehouse for each plan?

9. A football team plays in a large stadium. With a ticket price of \$20, the average attendance at recent games has been 30,000. A market survey indicates that for each \$1 increase in the ticket price, attendance decreases by 500.
   a. Express the number of spectators at a football game, N, as a function of the ticket price, x.
   b. Express the revenue from a football game, R, as a function of the ticket price, x.

10. A baseball team plays in a large stadium. With a ticket price of \$15, the average attendance at recent games has been 20,000. A market survey indicates that for each \$1 increase in the ticket price, attendance decreases by 400.
   a. Express the number of spectators at a baseball game, N, as a function of the ticket price, x.
   b. Express the revenue from a baseball game, R, as a function of the ticket price, x.

11. On a certain route, an airline carries 9000 passengers per month, each paying \$150. A market survey indicates that for each \$1 decrease in the ticket price, the airline will gain 50 passengers.
   a. Express the number of passengers per month, N, as a function of the ticket price, x.
   b. Express the monthly revenue for the route, R, as a function of the ticket price, x.

12. On a certain route, an airline carries 7000 passengers per month, each paying \$90. A market survey indicates that for each \$1 decrease in the ticket price, the airline will gain 60 passengers.
   a. Express the number of passengers per month, N, as a function of the ticket price, x.
   b. Express the monthly revenue for the route, R, as a function of the ticket price, x.

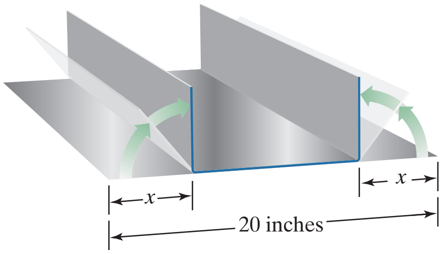
13. The annual yield per lemon tree is fairly constant at 320 pounds per tree when the number of trees per acre is 50 or fewer. For each additional tree over 50, the annual yield per tree for all trees on the acre decreases by 4 pounds due to overcrowding.
   a. Express the yield per tree, Y, in pounds, as a function of the number of lemon trees per acre, x.
   b. Express the total yield for an acre, T, in pounds, as a function of the number of lemon trees per acre, x.

14. The annual yield per orange tree is fairly constant at 270 pounds per tree when the number of trees per acre is 30 or fewer. For each additional tree over 30, the annual yield per tree for all trees on the acre decreases by 3 pounds due to overcrowding.
   a. Express the yield per tree, Y, in pounds, as a function of the number of orange trees per acre, x.
   b. Express the total yield for an acre, T, in pounds, as a function of the number of orange trees per acre, x.

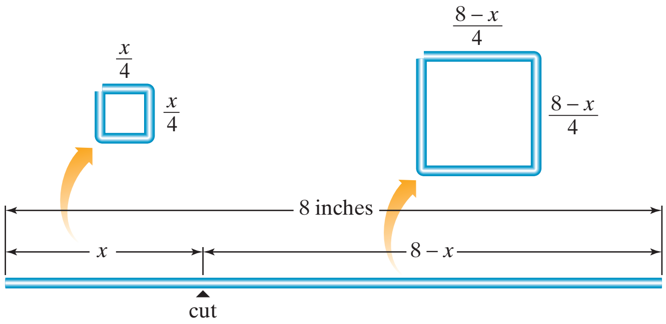
15. An open box is made from a square piece of cardboard 24 inches on a side by cutting identical squares from the corners and turning up the sides.
   a. Express the volume of the box, V, as a function of the length of the side of the square cut from each corner, x.
   b. Find and interpret $V(2), V(3), V(4), V(5)$, and $V(6)$. What is happening to the volume of the box as the length of the side of the square cut from each corner increases?
   c. Find the domain of V.

16. An open box is made from a square piece of cardboard 30 inches on a side by cutting identical squares from the corners and turning up the sides.
   a. Express the volume of the box, V, as a function of the length of the side of the square cut from each corner, x.
   b. Find and interpret $V(3), V(4), V(5), V(6)$, and $V(7)$. What is happening to the volume of the box as the length of the side of the square cut from each corner increases?
   c. Find the domain of V.

17. A rain gutter is made from sheets of aluminum that are 20 inches wide. As shown in the figure, the edges are turned up to form right angles. Express the cross-sectional area of the gutter, A, as a function of its depth, x.



18. A piece of wire is 8 inches long. The wire is cut into two pieces and then each piece is bent into a square. Express the sum of the areas of these squares, A, as a function of the length of the cut, x.



19. The sum of two numbers is 66. Express the product of the numbers, P, as a function of one of the numbers, x.

20. The sum of two numbers is 50. Express the product of the numbers, P, as a function of one of the numbers, x.

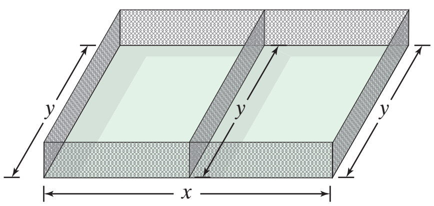
21. You have 800 feet of fencing to enclose a rectangular field. Express the area of the field, A, as a function of one of its dimensions, x.

22. You have 600 feet of fencing to enclose a rectangular field. Express the area of the field, A, as a function of one of its dimensions, x.

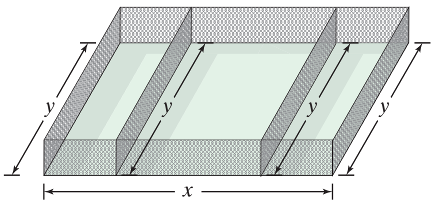
23. As in Exercise 21, you have 800 feet of fencing to enclose a rectangular field. However, one side of the field lies along a canal and requires no fencing. Express the area of the field, A, as a function of one of its dimensions, x.

24. As in Exercise 22, you have 600 feet of fencing to enclose a rectangular field. However, one side of the field lies along a canal and requires no fencing. Express the area of the field, A, as a function of one of its dimensions, x.

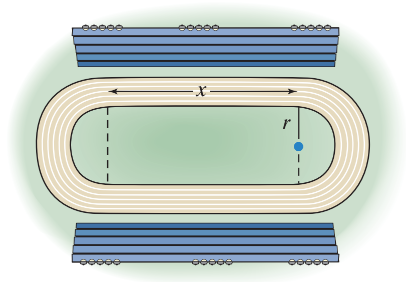
25. You have 1000 feet of fencing to enclose a rectangular playground and subdivide it into two smaller playgrounds by placing the fencing parallel to one of the sides. Express the area of the playground, A, as a function of one of its dimensions, x.



26. You have 1200 feet of fencing to enclose a rectangular region and subdivide it into three smaller rectangular regions by placing two fences parallel to one of the sides. Express the area of the enclosed rectangular region, A, as a function of one of its dimensions, x.



27. A new running track is to be constructed in the shape of a rectangle with semicircles at each end. The track is to be 440 yards long. Express the area of the region enclosed by the track, A, as a function of its radius, r.

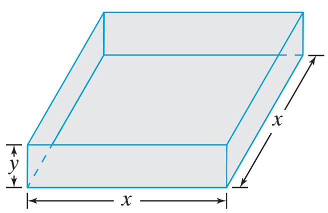


28. Work Exercise 27 if the length of the track is increased to 880 yards.

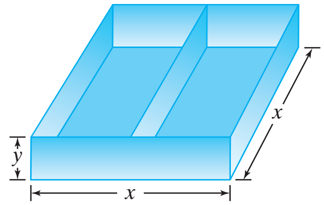
29. A contractor is to build a warehouse whose rectangular floor will have an area of 4000 square feet. The warehouse will be separated into two rectangular rooms by an interior wall. The cost of the exterior walls is \$175 per linear foot and the cost of the interior wall is \$125 per linear foot. Express the contractor's cost for building the walls, C, as a function of one of the dimensions of the warehouse's rectangular floor, x.

30. The area of a rectangular garden is 125 square feet. The garden is to be enclosed on three sides by a brick wall costing \$20 per foot and on one side by a fence costing \$9 per foot. Express the cost to enclose the garden, C, as a function of one of its dimensions, x.

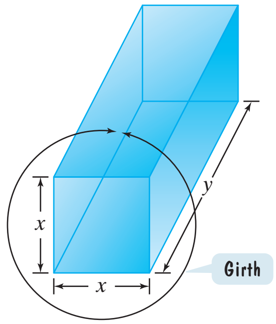
31. The figure shows an open box with a square base. The box is to have a volume of 10 cubic feet. Express the amount of material needed to construct the box, A, as a function of the length of a side of its square base, x.



32. The figure shows an open box with a square base and a partition down the middle. The box is to have a volume of 400 cubic inches. Express the amount of material needed to construct the box, A, as a function of the length of a side of its square base, x.



33. The figure shows a package whose front is a square. The length plus girth (the distance around) of the package is 300 inches. (This is the maximum length plus girth permitted by Federal Express for its overnight service.) Express the volume of the package, V, as a function of the length of a side of its square front, x.



34. Work Exercise 33 if the length plus girth of the package is 108 inches.

35. Your grandmother needs your help. She has \$50,000 to invest. Part of this money is to be invested in noninsured bonds paying 15% annual interest. The rest of this money is to be invested in a government-insured certificate of deposit paying 7% annual interest.

a. Express the interest from both investments, I, as a function of the amount of money invested in noninsured bonds, x.

b. Your grandmother told you that she requires \$6000 per year in extra income from both these investments. How much money should be placed in each investment?

36. You inherit \$18,750 with the stipulation that for the first year the money must be placed in two investments expected to pay 10% and 12% annual interest.

a. Express the expected interest from both investments, I, as a function of the amount of money invested at 10%, x.

b. If the total interest earned for the year was \$2117, how much money was invested at each rate?

37. You invested \$8000, part of it in a stock that paid 12% annual interest. However, the rest of the money suffered a 5% loss. Express the total annual income from both investments, I, as a function of the amount invested in the 12% stock, x.

38. You invested \$12,000, part of it in a stock that paid 14% annual interest. However, the rest of the money suffered a 6% loss. Express the total annual income from both investments, I, as a function of the amount invested in the 14% stock, x.

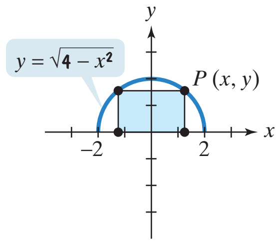
39. Let $P(x, y)$ be a point on the graph of $y = x^2 - 4$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

40. Let $P(x, y)$ be a point on the graph of $y = x^2 - 8$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

41. Let $P(x, y)$ be a point on the graph of $y = \sqrt{x}$. Express the distance, d, from P to $(1, 0)$ as a function of the point's x-coordinate.

42. Let $P(x, y)$ be a point on the graph of $y = \sqrt{x}$. Express the distance, d, from P to $(2, 0)$ as a function of the point's x-coordinate.

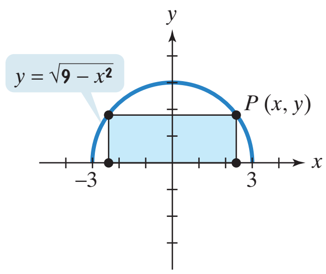
43. The figure shows a rectangle with two vertices on a semicircle of radius 2 and two vertices on the x-axis. Let $P(x, y)$ be the vertex that lies in the first quadrant.



a. Express the area of the rectangle, A, as a function of x.

b. Express the perimeter of the rectangle, P, as a function of x.

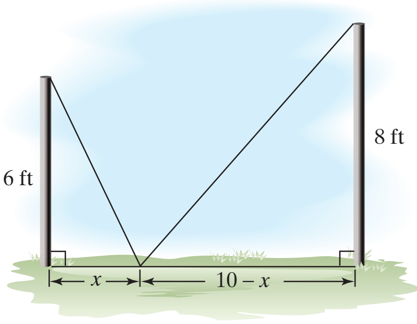
44. The figure shows a rectangle with two vertices on a semicircle of radius 3 and two vertices on the x-axis. Let $P(x, y)$ be the vertex that lies in the first quadrant.



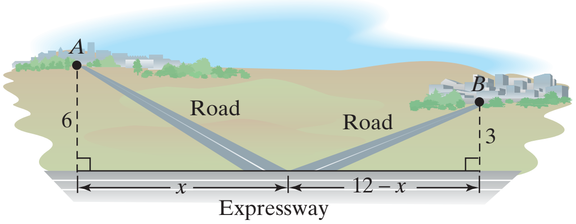
a. Express the area of the rectangle, A, as a function of x.

b. Express the perimeter of the rectangle, P, as a function of x.

45. Two vertical poles of length 6 feet and 8 feet, respectively, stand 10 feet apart. A cable reaches from the top of one pole to some point on the ground between the poles and then to the top of the other pole. Express the amount of cable used, f, as a function of the distance from the 6-foot pole to the point where the cable touches the ground, x.



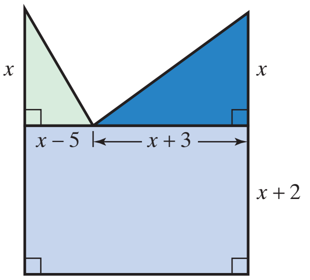
46. Towns A and B are located 6 miles and 3 miles, respectively, from a major expressway. The point on the expressway closest to town A is 12 miles from the point on the expressway closest to town B. Two new roads are to be built from A to the expressway and then to B. Express the combined lengths of the new roads, f, as a function of x as shown in the figure.



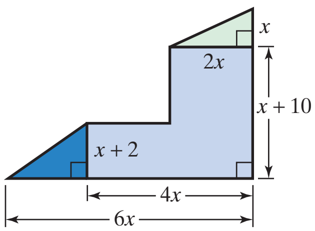
Practice Plus

In Exercises 47–48, express the area of each figure, A, as a function of one of its dimensions, x. Write the function's equation as a polynomial in standard form.

47.

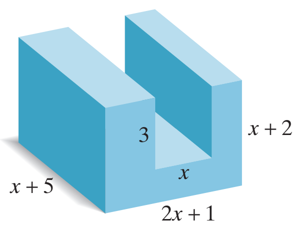


48.

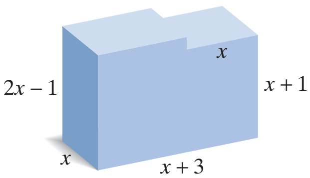


In Exercises 49–50, express the volume of each figure, V, as a function of one of its dimensions, x. Write the function's equation as a polynomial in standard form.

49.



50.



Writing in Mathematics

51. Throughout this section, we started with familiar formulas and created functions by substitution. Describe a specific situation in which we obtained a function using this technique.

52. Describe what should be displayed on the screen of a graphing utility to illustrate the solution that you obtained in Exercise 5(c) or Exercise 6(c).

53. In Exercise 9(b) or Exercise 10(b), describe what important information the team owners could learn from the revenue function.

54. In Exercise 13(b) or 14(b), describe what important information the growers could learn from the total-yield function.

55. In Exercise 31 or 32, describe what important information the box manufacturer could learn from the surface area function.

56. In calculus, you will learn powerful tools that reveal how functions behave. However, before applying these tools, there will be situations in which you are first required to obtain these functions from verbal descriptions. This is why your work in this section is so important. Because there is no rigid step-by-step procedure for modeling from verbal conditions, you might have had some difficulties obtaining functions for the assigned exercises. Discuss what you did if this happened to you. Did your course of action enhance your ability to model with functions?

Technology Exercise

57. Use a graphing utility to graph the function that you obtained in Exercise 1 or Exercise 2. Then use the TRACE or ZOOM feature to verify your answer in part (b) of the exercise.

58. Use a graphing utility to graph the two functions, f and g, that you obtained in any one exercise from Exercises 5–8. Then use the TRACE or INTERSECTION feature to verify your answer in part (c) of the exercise.

59. Use a graphing utility to graph the volume-of-the-box function, V, that you obtained in Exercise 15 or Exercise 16. Then use the TRACE or maximum function feature to find the length of the side of the square that should be cut from each corner of the cardboard to create a box with the greatest possible volume. What is the maximum volume of the open box?

60. Use a graphing utility to graph the area function, A, that you obtained in Exercise 21 or Exercise 22. Then use an appropriate feature on your graphing utility to find the dimensions of the field that result in the greatest possible area. What is the maximum area?

61. Use a graphing utility to graph the area function, A, that you obtained in Exercise 25 or Exercise 26. Then use an appropriate feature on your graphing utility to find the dimensions that result in the greatest possible area. What is the maximum area?

62. Use the maximum or minimum function feature of a graphing utility to provide useful numerical information to any one of the following: the manufacturer of the rain gutters in Exercise 17; the person enclosing the playground in Exercise 25; the contractor in Exercise 29; the manufacturer of the cylindrical cans in Check Point 5 (page 259).

Critical Thinking Exercises

Make Sense? *In Exercises 63–66, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

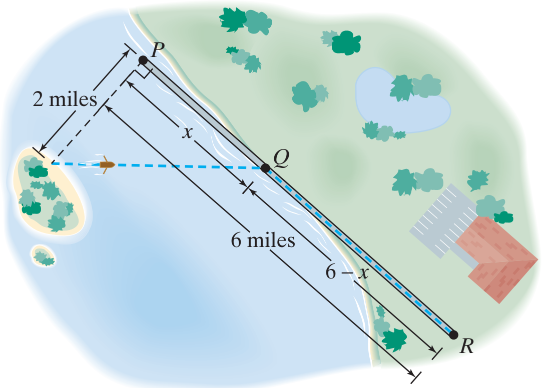
63. The function $f(x) = 30x + 0.08$ is a reasonable model for the monthly cost, f, in dollars, for a long-distance telephone plan in terms of the number of minutes of long-distance calls, x.

64. For each \$1 increase in the price of a \$300 plane ticket, an airline will lose 60 passengers, so if the ticket price is increased to $\$x$, the decrease in passengers is modeled by $60(300-x)$.

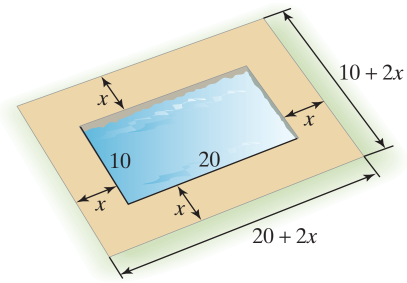
65. I know the perimeter of a rectangle, so I also know its area.

66. I encountered a number of problems where I had to solve an equation for a variable in order to express a function's equation in one variable.

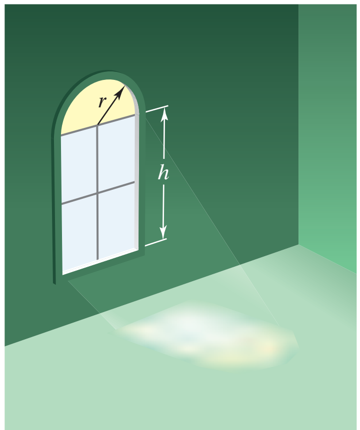
67. You are on an island 2 miles from the nearest point P on a straight shoreline, as shown in the figure. Six miles down the shoreline from point P is a restaurant, shown as point R. To reach the restaurant, you first row from the island to point Q, averaging 2 miles per hour. Then you jog the distance from Q to R, averaging 5 miles per hour. Express the time, T, it takes to go from the island to the restaurant as a function of the distance, x, from P, where you land the boat.



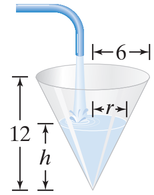
68. A pool measuring 20 meters by 10 meters is surrounded by a path of uniform width, as shown in the figure. Express the area of the path, A, in square meters, as a function of its width, x, in meters.



69. The figure shows a Norman window that has the shape of a rectangle with a semicircle attached at the top. The diameter of the semicircle is equal to the width of the rectangle. The window has a perimeter of 12 feet. Express the area of the window, A, as a function of its radius, r.



70. The figure shows water running into a container in the shape of a cone. The radius of the cone is 6 feet and its height is 12 feet. Express the volume of the water in the cone, V, as a function of the height of the water, h.



Preview Exercises

Exercises 71–73 will help you prepare for the material covered in the first section of the next chapter.

71. Multiply: $(7 - 3x)(-2 - 5x)$.

72. Simplify: $\sqrt{18} - \sqrt{8}$.

73. Rationalize the denominator: $\dfrac{7 + 4\sqrt{2}}{2 - 5\sqrt{2}}$.