Section 1.7 Combinations of Functions; Composite Functions

Objectives

1. Find the domain of a function.
2. Combine functions using the algebra of functions, specifying domains.
3. Form composite functions.
4. Determine domains for composite functions.
5. Write functions as compositions.

We're born. We die. **Figure 1.61** quantifies these statements by showing the numbers of births and deaths in the United States for six selected years.

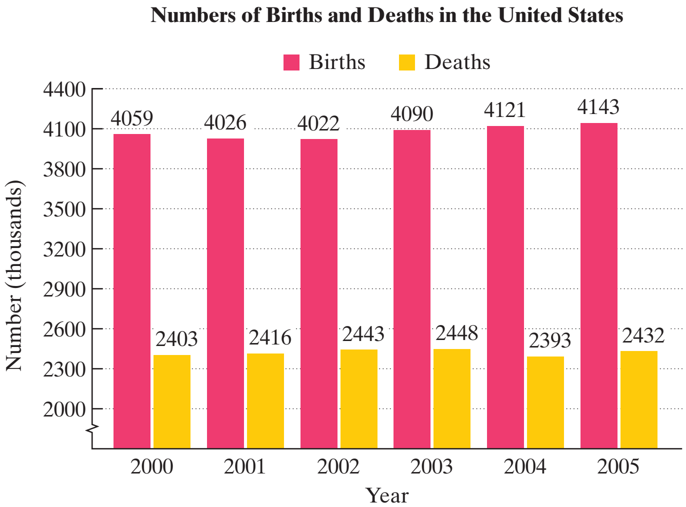


Figure 1.61
Source: U.S. Department of Health and Human Services

Study Tip

Throughout this section, we will be using the intersection of sets expressed in interval notation. Recall that the intersection of sets A and B, written $A \cap B$, is the set of elements common to both set A and set B. When sets A and B are in interval notation, to find the intersection, graph each interval and take the portion of the number line that the two graphs have in common. We will also be using notation involving the union of sets A and B, $A \cup B$, meaning the set of elements in A or in B or in both. For more detail, see Section P.1, pages 5–6 and Section P.9, pages 116–117.

In this section, we look at these data from the perspective of functions. By considering the yearly change in the U.S. population, you will see that functions can be subtracted using procedures that will remind you of combining algebraic expressions.

1 Find the domain of a function.

The Domain of a Function

We begin with two functions that model the data in **Figure 1.61**.

$$B(x) = 7.4x^2 - 15x + 4046 \qquad D(x) = -3.5x^2 + 20x + 2405$$

Number of births, $B(x)$, in thousands, x years after 2000

Number of deaths, $D(x)$, in thousands, x years after 2000

The years in **Figure 1.61** extend from 2000 through 2005. Because x represents the number of years after 2000,

$$\text{Domain of } B = \{x \mid x = 0, 1, 2, 3, 4, 5\}$$

and

$$\text{Domain of } D = \{x \mid x = 0, 1, 2, 3, 4, 5\}.$$

Functions that model data often have their domains explicitly given with the function's equation. However, for most functions, only an equation is given and the domain is not specified. In cases like this, the domain of a function f is the largest set of real numbers for which the value of $f(x)$ is a real number. For example, consider the function

$$f(x) = \frac{1}{x - 3}.$$

Because division by 0 is undefined, the denominator, $x - 3$, cannot be 0. Thus, x cannot equal 3. The domain of the function consists of all real numbers other than 3, represented by

$$\text{Domain of } f = \{x \mid x \text{ is a real number and } x \neq 3\}.$$

Using interval notation,

$$\text{Domain of } f = \underbrace{(-\infty, 3)}_{\text{All real numbers less than 3}} \underbrace{\cup}_{\text{or}} \underbrace{(3, \infty)}_{\text{All real numbers greater than 3}}.$$

Now consider a function involving a square root:

$$g(x) = \sqrt{x - 3}.$$

Because only nonnegative numbers have square roots that are real numbers, the expression under the square root sign, $x - 3$, must be nonnegative. We can use inspection to see that $x - 3 \geq 0$ if $x \geq 3$. The domain of g consists of all real numbers that are greater than or equal to 3:

$$\text{Domain of } g = \{x \mid x \geq 3\} \text{ or } [3, \infty).$$

Finding a Function's Domain

If a function f does not model data or verbal conditions, its domain is the largest set of real numbers for which the value of $f(x)$ is a real number. Exclude from a function's domain real numbers that cause division by zero and real numbers that result in an even root of a negative number.

EXAMPLE 1 Finding the Domain of a Function

Find the domain of each function.

a. $f(x) = x^2 - 7x$ **b.** $g(x) = \dfrac{3x + 2}{x^2 - 2x - 3}$ **c.** $h(x) = \sqrt{3x + 12}$.

Solution The domain is the set of all real numbers, $(-\infty, \infty)$, unless x appears in a denominator or a square root.

a. The function $f(x) = x^2 - 7x$ contains neither division nor a square root. For every real number, x, the algebraic expression $x^2 - 7x$ represents a real number. Thus, the domain of f is the set of all real numbers.

$$\text{Domain of } f = (-\infty, \infty)$$

b. The function $g(x) = \dfrac{3x + 2}{x^2 - 2x - 3}$ contains division. Because division by 0 is undefined, we must exclude from the domain the values of x that cause the denominator, $x^2 - 2x - 3$, to be 0. We can identify these values by setting $x^2 - 2x - 3$ equal to 0.

$$x^2 - 2x - 3 = 0 \quad \text{Set the function's denominator equal to 0.}$$
$$(x + 1)(x - 3) = 0 \quad \text{Factor.}$$
$$x + 1 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{Set each factor equal to 0.}$$
$$x = -1 \qquad\qquad x = 3 \quad \text{Solve the resulting equations.}$$

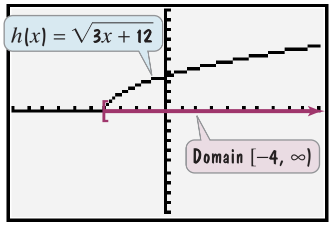
We must exclude -1 and 3 from the domain of g.

$$\text{Domain of } g = (-\infty, -1) \cup (-1, 3) \cup (3, \infty)$$

Study Tip

In parts (a) and (b), observe when to factor and when not to factor a polynomial.

- $f(x) = x^2 - 7x$
  Do not factor $x^2 - 7x$ and set it equal to zero. No values of x need be excluded from the domain.
- $g(x) = \dfrac{3x + 2}{x^2 - 2x - 3}$
  Do factor $x^2 - 2x - 3$ and set it equal to zero. We must exclude values of x that cause this denominator to be zero.



$[-10, 10, 1]$ by $[-10, 10, 1]$

Figure 1.62

c. The function $h(x) = \sqrt{3x + 12}$ contains an even root. Because only nonnegative numbers have real square roots, the quantity under the radical sign, $3x + 12$, must be greater than or equal to 0.

$$3x + 12 \geq 0 \quad \text{Set the function's radicand greater than or equal to 0.}$$

$$3x \geq -12 \quad \text{Subtract 12 from both sides.}$$

$$x \geq -4 \quad \text{Divide both sides by 3. Division by a positive number preserves the sense of the inequality.}$$

The domain of h consists of all real numbers greater than or equal to -4.

$$\text{Domain of } h = [-4, \infty)$$

The domain is highlighted on the x-axis in **Figure 1.62**.

Check Point 1 Find the domain of each function:

a. $f(x) = x^2 + 3x - 17$ **b.** $g(x) = \dfrac{5x}{x^2 - 49}$ **c.** $h(x) = \sqrt{9x - 27}$.

2 Combine functions using the algebra of functions, specifying domains.

The Algebra of Functions

We can combine functions using addition, subtraction, multiplication, and division by performing operations with the algebraic expressions that appear on the right side of the equations. For example, the functions $f(x) = 2x$ and $g(x) = x - 1$ can be combined to form the sum, difference, product, and quotient of f and g. Here's how it's done:

For each function, $f(x) = 2x$ and $g(x) = x - 1$.

Sum: $f + g$
$$(f + g)(x) = f(x) + g(x) = 2x + (x - 1) = 3x - 1$$

Difference: $f - g$
$$(f - g)(x) = f(x) - g(x) = 2x - (x - 1) = 2x - x + 1 = x + 1$$

Product: fg
$$(fg)(x) = f(x) \cdot g(x) = 2x(x - 1) = 2x^2 - 2x$$

Quotient: $\dfrac{f}{g}$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x}{x - 1}, \; x \neq 1.$$

The domain for each of these functions consists of all real numbers that are common to the domains of f and g. In the case of the quotient function $\dfrac{f(x)}{g(x)}$, we must remember not to divide by 0, so we add the further restriction that $g(x) \neq 0$.

The Algebra of Functions: Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions. The **sum** $f + g$, the **difference** $f - g$, the **product** fg, and the **quotient** $\dfrac{f}{g}$ are functions whose domains are the set of all real numbers common to the domains of f and g $(D_f \cap D_g)$, defined as follows:

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

EXAMPLE 2 Combining Functions

Let $f(x) = 2x - 1$ and $g(x) = x^2 + x - 2$. Find each of the following functions:

a. $(f + g)(x)$ **b.** $(f - g)(x)$ **c.** $(fg)(x)$ **d.** $\left(\frac{f}{g}\right)(x)$.

Determine the domain for each function.

Solution

a. $(f + g)(x) = f(x) + g(x)$ This is the definition of the sum $f + g$.

$= (2x - 1) + (x^2 + x - 2)$ Substitute the given functions.

$= x^2 + 3x - 3$ Remove parentheses and combine like terms.

b. $(f - g)(x) = f(x) - g(x)$ This is the definition of the difference $f - g$.

$= (2x - 1) - (x^2 + x - 2)$ Substitute the given functions.

$= 2x - 1 - x^2 - x + 2$ Remove parentheses and change the sign of each term in the second set of parentheses.

$= -x^2 + x + 1$ Combine like terms and arrange terms in descending powers of x.

c. $(fg)(x) = f(x) \cdot g(x)$ This is the definition of the product fg.

$= (2x - 1)(x^2 + x - 2)$ Substitute the given functions.

$= 2x(x^2 + x - 2) - 1(x^2 + x - 2)$ Multiply each term in the second factor by $2x$ and -1, respectively.

$= 2x^3 + 2x^2 - 4x - x^2 - x + 2$ Use the distributive property.

$= 2x^3 + (2x^2 - x^2) + (-4x - x) + 2$ Rearrange terms so that like terms are adjacent.

$= 2x^3 + x^2 - 5x + 2$ Combine like terms.

d. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ This is the definition of the quotient $\frac{f}{g}$.

$= \frac{2x - 1}{x^2 + x - 2}$ Substitute the given functions. This rational expression cannot be simplified.

Because the equations for f and g do not involve division or contain even roots, the domain of both f and g is the set of all real numbers. Thus, the domain of $f + g$, $f - g$, and fg is the set of all real numbers, $(-\infty, \infty)$.

The function $\frac{f}{g}$ contains division. We must exclude from its domain values of x that cause the denominator, $x^2 + x - 2$, to be 0. Let's identify these values.

$x^2 + x - 2 = 0$ Set the denominator of $\frac{f}{g}$ equal to 0.

$(x + 2)(x - 1) = 0$ Factor.

$x + 2 = 0$ or $x - 1 = 0$ Set each factor equal to 0.

$x = -2$ $\quad x = 1$ Solve the resulting equations.

We must exclude -2 and 1 from the domain of $\frac{f}{g}$.

$$\text{Domain of } \frac{f}{g} = (-\infty, -2) \cup (-2, 1) \cup (1, \infty)$$

Study Tip

If the function $\frac{f}{g}$ can be simplified, determine the domain *before* simplifying.

Example:

$f(x) = x^2 - 4$ and

$g(x) = x - 2$

$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 4}{x - 2}$

$x \neq 2$. The domain of $\frac{f}{g}$ is $(-\infty, 2) \cup (2, \infty)$.

$$= \frac{(x+2)\overset{1}{\cancel{(x-2)}}}{\underset{1}{\cancel{(x-2)}}} = x + 2$$

Check Point 2 Let $f(x) = x - 5$ and $g(x) = x^2 - 1$. Find each of the following functions:

a. $(f + g)(x)$ **b.** $(f - g)(x)$ **c.** $(fg)(x)$ **d.** $\left(\frac{f}{g}\right)(x)$.

Determine the domain for each function.

EXAMPLE 3 Adding Functions and Determining the Domain

Let $f(x) = \sqrt{x + 3}$ and $g(x) = \sqrt{x - 2}$. Find each of the following:

a. $(f + g)(x)$ **b.** the domain of $f + g$.

Solution

a. $(f + g)(x) = f(x) + g(x) = \sqrt{x + 3} + \sqrt{x - 2}$

b. The domain of $f + g$ is the set of all real numbers that are common to the domain of f and the domain of g. Thus, we must find the domains of f and g before finding their intersection.

- $f(x) = \sqrt{x + 3}$
  $x + 3$ must be nonnegative: $x + 3 \geq 0$. $D_f = [-3, \infty)$
- $g(x) = \sqrt{x - 2}$
  $x - 2$ must be nonnegative: $x - 2 \geq 0$. $D_g = [2, \infty)$

Now, we can use a number line to determine $D_f \cap D_g$, the domain of $f + g$. **Figure 1.63** shows the domain of f in blue and the domain of g in red. Can you see that all real numbers greater than or equal to 2 are common to both domains? This is shown in purple on the number line. Thus, the domain of $f + g$ is $[2, \infty)$.

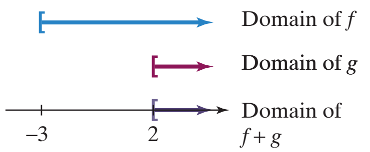
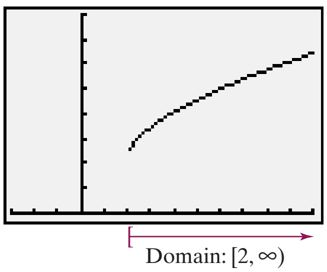


Figure 1.63 Finding the domain of the sum $f + g$

Technology

Graphic Connections



The graph on the left is the graph of

$$y = \sqrt{x + 3} + \sqrt{x - 2}$$

in a $[-3, 10, 1]$ by $[0, 8, 1]$ viewing rectangle. The graph reveals what we discovered algebraically in Example 3(b). The domain of this function is $[2, \infty)$.

Check Point 3 Let $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{x + 1}$. Find each of the following:

a. $(f + g)(x)$ **b.** the domain of $f + g$.

3 Form composite functions.

Composite Functions

There is another way of combining two functions. To help understand this new combination, suppose that your local computer store is having a sale. The models that are on sale cost either \$300 less than the regular price or 85% of the regular price. If x represents the computer's regular price, the discounts can be modeled with the following functions:

$$f(x) = x - 300 \qquad g(x) = 0.85x.$$

The computer is on sale for \$300 less than its regular price.

The computer is on sale for 85% of its regular price.

At the store, you bargain with the salesperson. Eventually, she makes an offer you can't refuse. The sale price will be 85% of the regular price followed by a \$300 reduction:

$$0.85x - 300.$$

85% of the regular price

followed by a \$300 reduction

In terms of the functions f and g, this offer can be obtained by taking the output of $g(x) = 0.85x$, namely $0.85x$, and using it as the input of f:

$$f(x) = x - 300$$

Replace x with $0.85x$, the output of $g(x) = 0.85x$.

$$f(0.85x) = 0.85x - 300.$$

Because $0.85x$ is $g(x)$, we can write this last equation as

$$f(g(x)) = 0.85x - 300.$$

We read this equation as "f of g of x is equal to $0.85x - 300$." We call $f(g(x))$ the **composition of the function f with g**, or a **composite function**. This composite function is written $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = 0.85x - 300.$$

This can be read "f of g of x" or "f composed with g of x."

Like all functions, we can evaluate $f \circ g$ for a specified value of x in the function's domain. For example, here's how to find the value of the composite function describing the offer you cannot refuse at 1400:

$$(f \circ g)(x) = 0.85x - 300$$

Replace x with 1400.

$$(f \circ g)(1400) = 0.85(1400) - 300 = 1190 - 300 = 890.$$

This means that a computer that regularly sells for \$1400 is on sale for \$890 subject to both discounts. We can use a partial table of coordinates for each of the discount functions, g and f, to numerically verify this result.

Computer's regular price

85% of the regular price

x	$g(x) = 0.85x$
1200	1020
1300	1105
1400	1190

85% of the regular price

\$300 reduction

x	$f(x) = x - 300$
1020	720
1105	805
1190	890

Computer's regular price / 85% of the regular price

x	$g(x) = 0.85x$
1200	1020
1300	1105
1400	1190

85% of the regular price / $300 reduction

x	$f(x) = x - 300$
1020	720
1105	805
1190	890

Tables for the discount functions (repeated)

Using these tables, we can find $(f \circ g)(1400)$:

$$(f \circ g)(1400) = f(g(1400)) = f(1190) = 890.$$

The table for g shows that $g(1400) = 1190$.

The table for f shows that $f(1190) = 890$.

This verifies that a computer that regularly sells for \$1400 is on sale for \$890 subject to both discounts.

Before you run out to buy a computer, let's generalize our discussion of the computer's double discount and define the composition of any two functions.

The Composition of Functions

The **composition of the function f with g** is denoted by $f \circ g$ and is defined by the equation

$$(f \circ g)(x) = f(g(x)).$$

The **domain of the composite function $f \circ g$** is the set of all x such that

1. x is in the domain of g and

2. $g(x)$ is in the domain of f.

The composition of f with g, $f \circ g$, is illustrated in **Figure 1.64**.

Step 1 Input x into g. **Step 2** Input $g(x)$ into f.

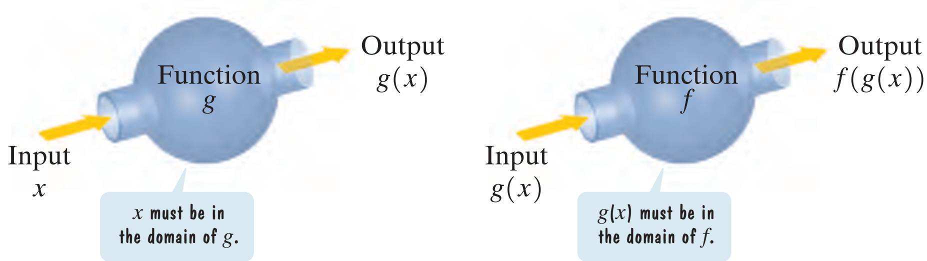


Figure 1.64

The figure reinforces the fact that the inside function g in $f(g(x))$ is done first.

EXAMPLE 4 Forming Composite Functions

Given $f(x) = 3x - 4$ and $g(x) = x^2 - 2x + 6$, find each of the following:

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(g \circ f)(1)$.

Solution

a. We begin with $(f \circ g)(x)$, the composition of f with g. Because $(f \circ g)(x)$ means $f(g(x))$, we must replace each occurrence of x in the equation for f with $g(x)$.

$$f(x) = 3x - 4$$ This is the given equation for *f*.

Replace x with $g(x)$.

$$(f \circ g)(x) = f(g(x)) = 3g(x) - 4$$

$$= 3(x^2 - 2x + 6) - 4$$ Because $g(x) = x^2 - 2x + 6$, replace $g(x)$ with $x^2 - 2x + 6$.

$$= 3x^2 - 6x + 18 - 4$$ Use the distributive property.

$$= 3x^2 - 6x + 14$$ Simplify.

Thus, $(f \circ g)(x) = 3x^2 - 6x + 14$.

b. Next, we find $(g \circ f)(x)$, the composition of g with f. Because $(g \circ f)(x)$ means $g(f(x))$, we must replace each occurrence of x in the equation for g with $f(x)$.

$$g(x) = x^2 - 2x + 6$$ This is the equation for *g*.

Replace x with $f(x)$.

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 - 2f(x) + 6$$

$$= (3x - 4)^2 - 2(3x - 4) + 6$$ Because $f(x) = 3x - 4$, replace $f(x)$ with $3x - 4$.

$$= 9x^2 - 24x + 16 - 6x + 8 + 6$$ Use $(A - B)^2 = A^2 - 2AB + B^2$ to square $3x - 4$.

$$= 9x^2 - 30x + 30$$ Simplify: $-24x - 6x = -30x$ and $16 + 8 + 6 = 30$.

Thus, $(g \circ f)(x) = 9x^2 - 30x + 30$. **Notice that $(f \circ g)(x)$ is not the same function as $(g \circ f)(x)$.**

c. We can use $(g \circ f)(x)$ to find $(g \circ f)(1)$.

$$(g \circ f)(x) = 9x^2 - 30x + 30$$

Replace x with 1.

$$(g \circ f)(1) = 9 \cdot 1^2 - 30 \cdot 1 + 30 = 9 - 30 + 30 = 9$$

It is also possible to find $(g \circ f)(1)$ without determining $(g \circ f)(x)$.

$$(g \circ f)(1) = g(f(1)) = g(-1) = 9$$

First find $f(1)$. $f(x) = 3x - 4$, so $f(1) = 3 \cdot 1 - 4 = -1$.

Next find $g(-1)$. $g(x) = x^2 - 2x + 6$, so $g(-1) = (-1)^2 - 2(-1) + 6 = 1 + 2 + 6 = 9$.

Check Point 4 Given $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find each of the following:

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(f \circ g)(-1)$.

4 Determine domains for composite functions.

We need to be careful in determining the domain for a composite function.

Excluding Values from the Domain of $(f \circ g)(x) = f(g(x))$

The following values must be excluded from the input x:

- If x is not in the domain of g, it must not be in the domain of $f \circ g$.
- Any x for which $g(x)$ is not in the domain of f must not be in the domain of $f \circ g$.

EXAMPLE 5 Forming a Composite Function and Finding Its Domain

Given $f(x) = \frac{2}{x-1}$ and $g(x) = \frac{3}{x}$, find each of the following:

a. $(f \circ g)(x)$ **b.** the domain of $f \circ g$.

Solution

a. Because $(f \circ g)(x)$ means $f(g(x))$, we must replace x in $f(x) = \frac{2}{x-1}$ with $g(x)$.

$$(f \circ g)(x) = f(g(x)) = \frac{2}{g(x)-1} = \frac{2}{\frac{3}{x}-1} = \frac{2}{\frac{3}{x}-1} \cdot \frac{x}{x} = \frac{2x}{3-x}$$

$g(x) = \frac{3}{x}$

Simplify the complex fraction by multiplying by $\frac{x}{x}$, or 1.

Study Tip

The procedure for simplifying complex fractions can be found in Section P.6, pages 75–77.

Thus, $(f \circ g)(x) = \frac{2x}{3-x}$.

b. We determine values to exclude from the domain of $(f \circ g)(x)$ in two steps.

Rules for Excluding Numbers from the Domain of $(f \circ g)(x) = f(g(x))$	Applying the Rules to $f(x) = \frac{2}{x-1}$ and $g(x) = \frac{3}{x}$
If x is not in the domain of g, it must not be in the domain of $f \circ g$.	Because $g(x) = \frac{3}{x}$, 0 is not in the domain of g. Thus, 0 must be excluded from the domain of $f \circ g$.
Any x for which $g(x)$ is not in the domain of f must not be in the domain of $f \circ g$.	Because $f(g(x)) = \frac{2}{g(x)-1}$, we must exclude from the domain of $f \circ g$ any x for which $g(x) = 1$. $\frac{3}{x} = 1$ Set $g(x)$ equal to 1. $3 = x$ Multiply both sides by x. 3 must be excluded from the domain of $f \circ g$.

We see that 0 and 3 must be excluded from the domain of $f \circ g$. The domain of $f \circ g$ is

$$(-\infty, 0) \cup (0, 3) \cup (3, \infty).$$

Check Point 5 Given $f(x) = \frac{4}{x+2}$ and $g(x) = \frac{1}{x}$, find each of the following:

a. $(f \circ g)(x)$ **b.** the domain of $f \circ g$.

5 Write functions as compositions.

Decomposing Functions

When you form a composite function, you "compose" two functions to form a new function. It is also possible to reverse this process. That is, you can "decompose" a given function and express it as a composition of two functions. Although there is more than one way to do this, there is often a "natural" selection that comes to mind first. For example, consider the function h defined by

$$h(x) = (3x^2 - 4x + 1)^5.$$

The function h takes $3x^2 - 4x + 1$ and raises it to the power 5. A natural way to write h as a composition of two functions is to raise the function $g(x) = 3x^2 - 4x + 1$ to the power 5. Thus, if we let

$$f(x) = x^5 \text{ and } g(x) = 3x^2 - 4x + 1, \text{ then}$$

$$(f \circ g)(x) = f(g(x)) = f(3x^2 - 4x + 1) = (3x^2 - 4x + 1)^5.$$

EXAMPLE 6 Writing a Function as a Composition

Express $h(x)$ as a composition of two functions:

$$h(x) = \sqrt[3]{x^2 + 1}.$$

Solution The function h takes $x^2 + 1$ and takes its cube root. A natural way to write h as a composition of two functions is to take the cube root of the function $g(x) = x^2 + 1$. Thus, we let

$$f(x) = \sqrt[3]{x} \text{ and } g(x) = x^2 + 1.$$

We can check this composition by finding $(f \circ g)(x)$. This should give the original function, namely $h(x) = \sqrt[3]{x^2 + 1}$.

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt[3]{x^2 + 1} = h(x)$$

Study Tip

Suppose the form of function h is $h(x) = (\text{algebraic expression})^{\text{power}}$. Function h can be expressed as a composition, $f \circ g$, using

$f(x) = x^{\text{power}}$

$g(x) = \text{algebraic expression}.$

Check Point 6 Express $h(x)$ as a composition of two functions:

$$h(x) = \sqrt{x^2 + 5}.$$

Exercise Set 1.7

Practice Exercises

In Exercises 1–30, find the domain of each function.

1. $f(x) = 3(x - 4)$

2. $f(x) = 2(x + 5)$

3. $g(x) = \dfrac{3}{x - 4}$

4. $g(x) = \dfrac{2}{x + 5}$

5. $f(x) = x^2 - 2x - 15$

6. $f(x) = x^2 + x - 12$

7. $g(x) = \dfrac{3}{x^2 - 2x - 15}$

8. $g(x) = \dfrac{2}{x^2 + x - 12}$

9. $f(x) = \dfrac{1}{x + 7} + \dfrac{3}{x - 9}$

10. $f(x) = \dfrac{1}{x + 8} + \dfrac{3}{x - 10}$

11. $g(x) = \dfrac{1}{x^2 + 1} - \dfrac{1}{x^2 - 1}$

12. $g(x) = \dfrac{1}{x^2 + 4} - \dfrac{1}{x^2 - 4}$

13. $h(x) = \dfrac{4}{\frac{3}{x} - 1}$

14. $h(x) = \dfrac{5}{\frac{4}{x} - 1}$

15. $f(x) = \dfrac{1}{\frac{4}{x - 1} - 2}$

16. $f(x) = \dfrac{1}{\frac{4}{x - 2} - 3}$

17. $f(x) = \sqrt{x - 3}$

18. $f(x) = \sqrt{x + 2}$

19. $g(x) = \dfrac{1}{\sqrt{x-3}}$

20. $g(x) = \dfrac{1}{\sqrt{x+2}}$

21. $g(x) = \sqrt{5x+35}$

22. $g(x) = \sqrt{7x-70}$

23. $f(x) = \sqrt{24-2x}$

24. $f(x) = \sqrt{84-6x}$

25. $h(x) = \sqrt{x-2} + \sqrt{x+3}$

26. $h(x) = \sqrt{x-3} + \sqrt{x+4}$

27. $g(x) = \dfrac{\sqrt{x-2}}{x-5}$

28. $g(x) = \dfrac{\sqrt{x-3}}{x-6}$

29. $f(x) = \dfrac{2x+7}{x^3-5x^2-4x+20}$

30. $f(x) = \dfrac{7x+2}{x^3-2x^2-9x+18}$

In Exercises 31–48, find $f+g, f-g, fg$, and $\frac{f}{g}$. Determine the domain for each function.

31. $f(x) = 2x+3, g(x) = x-1$
32. $f(x) = 3x-4, g(x) = x+2$
33. $f(x) = x-5, g(x) = 3x^2$
34. $f(x) = x-6, g(x) = 5x^2$
35. $f(x) = 2x^2-x-3, g(x) = x+1$
36. $f(x) = 6x^2-x-1, g(x) = x-1$
37. $f(x) = 3-x^2, g(x) = x^2+2x-15$
38. $f(x) = 5-x^2, g(x) = x^2+4x-12$
39. $f(x) = \sqrt{x}, g(x) = x-4$
40. $f(x) = \sqrt{x}, g(x) = x-5$
41. $f(x) = 2+\dfrac{1}{x}, g(x) = \dfrac{1}{x}$
42. $f(x) = 6-\dfrac{1}{x}, g(x) = \dfrac{1}{x}$
43. $f(x) = \dfrac{5x+1}{x^2-9}, g(x) = \dfrac{4x-2}{x^2-9}$
44. $f(x) = \dfrac{3x+1}{x^2-25}, g(x) = \dfrac{2x-4}{x^2-25}$
45. $f(x) = \sqrt{x+4}, g(x) = \sqrt{x-1}$
46. $f(x) = \sqrt{x+6}, g(x) = \sqrt{x-3}$
47. $f(x) = \sqrt{x-2}, g(x) = \sqrt{2-x}$
48. $f(x) = \sqrt{x-5}, g(x) = \sqrt{5-x}$

In Exercises 49–64, find

a. $(f \circ g)(x)$; **b.** $(g \circ f)(x)$; **c.** $(f \circ g)(2)$.

49. $f(x) = 2x, g(x) = x+7$
50. $f(x) = 3x, g(x) = x-5$
51. $f(x) = x+4, g(x) = 2x+1$
52. $f(x) = 5x+2, g(x) = 3x-4$
53. $f(x) = 4x-3, g(x) = 5x^2-2$
54. $f(x) = 7x+1, g(x) = 2x^2-9$
55. $f(x) = x^2+2, g(x) = x^2-2$
56. $f(x) = x^2+1, g(x) = x^2-3$
57. $f(x) = 4-x, g(x) = 2x^2+x+5$
58. $f(x) = 5x-2, g(x) = -x^2+4x-1$
59. $f(x) = \sqrt{x}, g(x) = x-1$
60. $f(x) = \sqrt{x}, g(x) = x+2$
61. $f(x) = 2x-3, g(x) = \dfrac{x+3}{2}$
62. $f(x) = 6x-3, g(x) = \dfrac{x+3}{6}$
63. $f(x) = \dfrac{1}{x}, g(x) = \dfrac{1}{x}$
64. $f(x) = \dfrac{2}{x}, g(x) = \dfrac{2}{x}$

In Exercises 65–72, find

a. $(f \circ g)(x)$; **b.** *the domain of* $f \circ g$.

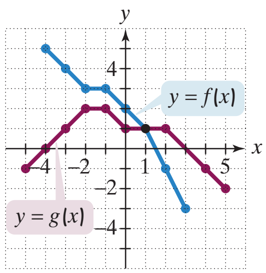
65. $f(x) = \dfrac{2}{x+3}, g(x) = \dfrac{1}{x}$
66. $f(x) = \dfrac{5}{x+4}, g(x) = \dfrac{1}{x}$
67. $f(x) = \dfrac{x}{x+1}, g(x) = \dfrac{4}{x}$
68. $f(x) = \dfrac{x}{x+5}, g(x) = \dfrac{6}{x}$
69. $f(x) = \sqrt{x}, g(x) = x-2$
70. $f(x) = \sqrt{x}, g(x) = x-3$
71. $f(x) = x^2+4, g(x) = \sqrt{1-x}$
72. $f(x) = x^2+1, g(x) = \sqrt{2-x}$

In Exercises 73–80, express the given function h as a composition of two functions f and g so that $h(x) = (f \circ g)(x)$.

73. $h(x) = (3x-1)^4$
74. $h(x) = (2x-5)^3$
75. $h(x) = \sqrt[3]{x^2-9}$
76. $h(x) = \sqrt{5x^2+3}$
77. $h(x) = |2x-5|$
78. $h(x) = |3x-4|$
79. $h(x) = \dfrac{1}{2x-3}$
80. $h(x) = \dfrac{1}{4x+5}$

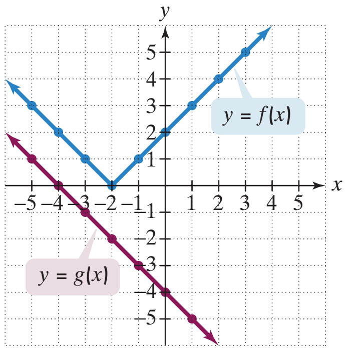
Practice Plus

Use the graphs of f and g to solve Exercises 81–88.



81. Find $(f+g)(-3)$.
82. Find $(g-f)(-2)$.
83. Find $(fg)(2)$.
84. Find $\left(\dfrac{g}{f}\right)(3)$.
85. Find the domain of $f+g$.
86. Find the domain of $\dfrac{f}{g}$.
87. Graph $f+g$.
88. Graph $f-g$.

In Exercises 89–92, use the graphs of f and g to evaluate each composite function.



89. $(f \circ g)(-1)$ **90.** $(f \circ g)(1)$

91. $(g \circ f)(0)$ **92.** $(g \circ f)(-1)$

In Exercises 93–94, find all values of x satisfying the given conditions.

93. $f(x) = 2x - 5$, $g(x) = x^2 - 3x + 8$, and $(f \circ g)(x) = 7$.

94. $f(x) = 1 - 2x$, $g(x) = 3x^2 + x - 1$, and $(f \circ g)(x) = -5$.

Application Exercises

We opened the section with functions that model the numbers of births and deaths in the United States from 2000 through 2005:

$$B(x) = 7.4x^2 - 15x + 4046 \qquad D(x) = -3.5x^2 + 20x + 2405.$$

Number of births, $B(x)$, in thousands, x years after 2000

Number of deaths, $D(x)$, in thousands, x years after 2000

Use these functions to solve Exercises 95–96.

95. **a.** Write a function that models the change in U.S. population for each year from 2000 through 2005.

b. Use the function from part (a) to find the change in U.S. population in 2003.

c. Does the result in part (b) overestimate or underestimate the actual population change in 2003 obtained from the data in **Figure 1.61** on page 220? By how much?

96. **a.** Write a function that models the total number of births and deaths in the United States for each year from 2000 through 2005.

b. Use the function from part (a) to find the total number of births and deaths in the United States in 2005.

c. Does the result in part (b) overestimate or underestimate the actual number of total births and deaths in 2005 obtained from the data in **Figure 1.61** on page 220? By how much?

97. A company that sells radios has yearly fixed costs of \$600,000. It costs the company \$45 to produce each radio. Each radio will sell for \$65. The company's costs and revenue are modeled by the following functions, where x represents the number of radios produced and sold:

$C(x) = 600{,}000 + 45x$ This function models the company's costs.

$R(x) = 65x.$ This function models the company's revenue.

Find and interpret $(R - C)(20{,}000)$, $(R - C)(30{,}000)$, and $(R - C)(40{,}000)$.

98. A department store has two locations in a city. From 2004 through 2008, the profits for each of the store's two branches are modeled by the functions $f(x) = -0.44x + 13.62$ and $g(x) = 0.51x + 11.14$. In each model, x represents the number of years after 2004, and f and g represent the profit, in millions of dollars.

a. What is the slope of f? Describe what this means.

b. What is the slope of g? Describe what this means.

c. Find $f + g$. What is the slope of this function? What does this mean?

99. The regular price of a computer is x dollars. Let $f(x) = x - 400$ and $g(x) = 0.75x$.

a. Describe what the functions f and g model in terms of the price of the computer.

b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the computer.

c. Repeat part (b) for $(g \circ f)(x)$.

d. Which composite function models the greater discount on the computer, $f \circ g$ or $g \circ f$? Explain.

100. The regular price of a pair of jeans is x dollars. Let $f(x) = x - 5$ and $g(x) = 0.6x$.

a. Describe what functions f and g model in terms of the price of the jeans.

b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the jeans.

c. Repeat part (b) for $(g \circ f)(x)$.

d. Which composite function models the greater discount on the jeans, $f \circ g$ or $g \circ f$? Explain.

Writing in Mathematics

101. If a function is defined by an equation, explain how to find its domain.

102. If equations for f and g are given, explain how to find $f - g$.

103. If equations for two functions are given, explain how to obtain the quotient function and its domain.

104. Describe a procedure for finding $(f \circ g)(x)$. What is the name of this function?

105. Describe the values of x that must be excluded from the domain of $(f \circ g)(x)$.

Technology Exercises

106. Graph $y_1 = x^2 - 2x$, $y_2 = x$, and $y_3 = y_1 \div y_2$ in the same $[-10, 10, 1]$ by $[-10, 10, 1]$ viewing rectangle. Then use the $\boxed{\text{TRACE}}$ feature to trace along y_3. What happens at $x = 0$? Explain why this occurs.

107. Graph $y_1 = \sqrt{2 - x}$, $y_2 = \sqrt{x}$, and $y_3 = \sqrt{2 - y_2}$ in the same $[-4, 4, 1]$ by $[0, 2, 1]$ viewing rectangle. If y_1 represents f and y_2 represents g, use the graph of y_3 to find the domain of $f \circ g$. Then verify your observation algebraically.

Critical Thinking Exercises

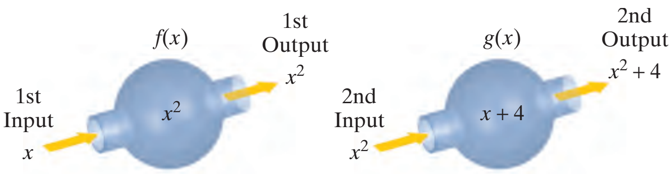
Make Sense? *In Exercises 108–111, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

108. I used a function to model data from 1980 through 2005. The independent variable in my model represented the number of years after 1980, so the function's domain was $\{x \mid x = 0, 1, 2, 3, \ldots, 25\}$.

109. I have two functions. Function f models total world population x years after 2000 and function g models population of the world's more-developed regions x years after 2000. I can use $f - g$ to determine the population of the world's less-developed regions for the years in both function's domains.

110. I must have made a mistake in finding the composite functions $f \circ g$ and $g \circ f$, because I notice that $f \circ g$ is not the same function as $g \circ f$.

111. This diagram illustrates that $f(g(x)) = x^2 + 4$.



In Exercises 112–115, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

112. If $f(x) = x^2 - 4$ and $g(x) = \sqrt{x^2 - 4}$, then $(f \circ g)(x) = -x^2$ and $(f \circ g)(5) = -25$.

113. There can never be two functions f and g, where $f \neq g$, for which $(f \circ g)(x) = (g \circ f)(x)$.

114. If $f(7) = 5$ and $g(4) = 7$, then $(f \circ g)(4) = 35$.

115. If $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$, then $(f \circ g)(5) = g(2)$.

116. Prove that if f and g are even functions, then fg is also an even function.

117. Define two functions f and g so that $f \circ g = g \circ f$.

Preview Exercises

Exercises 118–120 will help you prepare for the material covered in the next section.

118. Consider the function defined by

$$\{(-2, 4), (-1, 1), (1, 1), (2, 4)\}.$$

Reverse the components of each ordered pair and write the resulting relation. Is this relation a function?

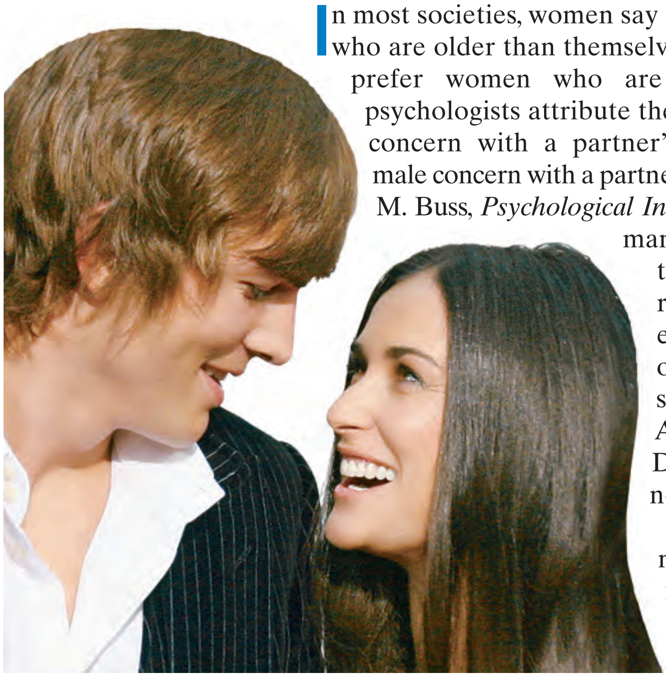
119. Solve for y: $x = \dfrac{5}{y} + 4$.

120. Solve for y: $x = y^2 - 1, y \geq 0$.

Section 1.8 Inverse Functions

Objectives

1. Verify inverse functions.
2. Find the inverse of a function.
3. Use the horizontal line test to determine if a function has an inverse function.
4. Use the graph of a one-to-one function to graph its inverse function.
5. Find the inverse of a function and graph both functions on the same axes.



In most societies, women say they prefer to marry men who are older than themselves, whereas men say they prefer women who are younger. Evolutionary psychologists attribute these preferences to female concern with a partner's material resources and male concern with a partner's fertility (*Source:* David M. Buss, *Psychological Inquiry*, 6, 1–30). When the man is considerably older than the woman, people rarely comment. However, when the woman is older, as in the relationship between actors Ashton Kutcher and Demi Moore, people take notice.

Figure 1.65 on the next page shows the preferred age difference in a mate in five selected countries.