134. Graph each of the following functions in the same viewing rectangle and then place the functions in order from the one that increases most slowly to the one that increases most rapidly.

$$y = x, y = \sqrt{x}, y = e^x, y = \ln x, y = x^x, y = x^2$$

Critical Thinking Exercises

Make Sense? *In Exercises 135–138, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

135. I've noticed that exponential functions and logarithmic functions exhibit inverse, or opposite, behavior in many ways. For example, a vertical translation shifts an exponential function's horizontal asymptote and a horizontal translation shifts a logarithmic function's vertical asymptote.

136. I estimate that $\log_8 16$ lies between 1 and 2 because $8^1 = 8$ and $8^2 = 64$.

137. I can evaluate some common logarithms without having to use a calculator.

138. An earthquake of magnitude 8 on the Richter scale is twice as intense as an earthquake of magnitude 4.

In Exercises 139–142, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

139. $\dfrac{\log_2 8}{\log_2 4} = \dfrac{8}{4}$

140. $\log(-100) = -2$

141. The domain of $f(x) = \log_2 x$ is $(-\infty, \infty)$.

142. $\log_b x$ is the exponent to which b must be raised to obtain x.

143. Without using a calculator, find the exact value of

$$\frac{\log_3 81 - \log_\pi 1}{\log_{2\sqrt{2}} 8 - \log 0.001}.$$

144. Without using a calculator, find the exact value of $\log_4[\log_3(\log_2 8)]$.

145. Without using a calculator, determine which is the greater number: $\log_4 60$ or $\log_3 40$.

Group Exercise

146. This group exercise involves exploring the way we grow. Group members should create a graph for the function that models the percentage of adult height attained by a boy who is x years old, $f(x) = 29 + 48.8\log(x + 1)$. Let $x = 1, 2, 3, \ldots, 12$, find function values, and connect the resulting points with a smooth curve. Then create a graph for the function that models the percentage of adult height attained by a girl who is x years old, $g(x) = 62 + 35\log(x - 4)$. Let $x = 5, 6, 7, \ldots, 15$, find function values, and connect the resulting points with a smooth curve. Group members should then discuss similarities and differences in the growth patterns for boys and girls based on the graphs.

Preview Exercises

Exercises 147–149 will help you prepare for the material covered in the next section. In each exercise, evaluate the indicated logarithmic expressions without using a calculator.

147. **a.** Evaluate: $\log_2 32$.
b. Evaluate: $\log_2 8 + \log_2 4$.
c. What can you conclude about $\log_2 32$, or $\log_2(8 \cdot 4)$?

148. **a.** Evaluate: $\log_2 16$.
b. Evaluate: $\log_2 32 - \log_2 2$.
c. What can you conclude about

$$\log_2 16, \text{ or } \log_2\left(\frac{32}{2}\right)?$$

149. **a.** Evaluate: $\log_3 81$.
b. Evaluate: $2\log_3 9$.
c. What can you conclude about

$$\log_3 81, \text{ or } \log_3 9^2?$$

Section 3.3 Properties of Logarithms

Objectives

1. Use the product rule.
2. Use the quotient rule.
3. Use the power rule.
4. Expand logarithmic expressions.
5. Condense logarithmic expressions.
6. Use the change-of-base property.

We all learn new things in different ways. In this section, we consider important properties of logarithms. What would be the most effective way for you to learn these properties? Would it be helpful to use your graphing utility and discover one of these properties for yourself? To do so, work Exercise 133 in Exercise Set 3.2 before continuing. Would it be helpful to evaluate certain logarithmic expressions that suggest three of the properties? If this is the case, work Preview Exercises 147–149 in Exercise Set 3.2 before continuing. Would the properties become more meaningful if you could see exactly where they come from? If so, you will find details of the proofs of many of these properties in Appendix A. The remainder of our work in this chapter will be based on the properties of logarithms that you learn in this section.

1 Use the product rule.

The Product Rule

Properties of exponents correspond to properties of logarithms. For example, when we multiply with the same base, we add exponents:

$$b^m \cdot b^n = b^{m+n}.$$

This property of exponents, coupled with an awareness that a logarithm is an exponent, suggests the following property, called the **product rule**:

Discovery

We know that $\log 100{,}000 = 5$. Show that you get the same result by writing 100,000 as $1000 \cdot 100$ and then using the product rule. Then verify the product rule by using other numbers whose logarithms are easy to find.

The Product Rule

Let b, M, and N be positive real numbers with $b \neq 1$.

$$\log_b(MN) = \log_b M + \log_b N$$

The logarithm of a product is the sum of the logarithms.

When we use the product rule to write a single logarithm as the sum of two logarithms, we say that we are **expanding a logarithmic expression**. For example, we can use the product rule to expand $\ln(7x)$:

$$\ln(7x) = \ln 7 + \ln x.$$

The logarithm of a product | is | the sum of the logarithms.

EXAMPLE 1 Using the Product Rule

Use the product rule to expand each logarithmic expression:

a. $\log_4(7 \cdot 5)$ **b.** $\log(10x)$.

Solution

a. $\log_4(7 \cdot 5) = \log_4 7 + \log_4 5$ The logarithm of a product is the sum of the logarithms.

b. $\log(10x) = \log 10 + \log x$ The logarithm of a product is the sum of the logarithms. These are common logarithms with base 10 understood.

$= 1 + \log x$ Because $\log_b b = 1$, then $\log 10 = 1$.

Check Point 1 Use the product rule to expand each logarithmic expression:

a. $\log_6(7 \cdot 11)$ **b.** $\log(100x)$.

2 Use the quotient rule.

The Quotient Rule

When we divide with the same base, we subtract exponents:

$$\frac{b^m}{b^n} = b^{m-n}.$$

This property suggests the following property of logarithms, called the **quotient rule**:

Discovery

We know that $\log_2 16 = 4$. Show that you get the same result by writing 16 as $\frac{32}{2}$ and then using the quotient rule. Then verify the quotient rule using other numbers whose logarithms are easy to find.

The Quotient Rule

Let b, M, and N be positive real numbers with $b \neq 1$.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

The logarithm of a quotient is the difference of the logarithms.

When we use the quotient rule to write a single logarithm as the difference of two logarithms, we say that we are **expanding a logarithmic expression**. For example, we can use the quotient rule to expand $\log \frac{x}{2}$:

$$\log\left(\frac{x}{2}\right) = \log x - \log 2.$$

The logarithm of a quotient | is | the difference of the logarithms.

EXAMPLE 2 Using the Quotient Rule

Use the quotient rule to expand each logarithmic expression:

a. $\log_7\left(\frac{19}{x}\right)$ **b.** $\ln\left(\frac{e^3}{7}\right)$.

Solution

a. $\log_7\left(\frac{19}{x}\right) = \log_7 19 - \log_7 x$ The logarithm of a quotient is the difference of the logarithms.

b. $\ln\left(\frac{e^3}{7}\right) = \ln e^3 - \ln 7$ The logarithm of a quotient is the difference of the logarithms. These are natural logarithms with base e understood.

$= 3 - \ln 7$ Because $\ln e^x = x$, then $\ln e^3 = 3$.

Check Point 2 Use the quotient rule to expand each logarithmic expression:

a. $\log_8\left(\frac{23}{x}\right)$ **b.** $\ln\left(\frac{e^5}{11}\right)$.

3 Use the power rule.

The Power Rule

When an exponential expression is raised to a power, we multiply exponents:

$$(b^m)^n = b^{mn}.$$

This property suggests the following property of logarithms, called the **power rule**:

The Power Rule

Let b and M be positive real numbers with $b \neq 1$, and let p be any real number.

$$\log_b M^p = p \log_b M$$

The logarithm of a number with an exponent is the product of the exponent and the logarithm of that number.

When we use the power rule to "pull the exponent to the front," we say that we are **expanding a logarithmic expression**. For example, we can use the power rule to expand $\ln x^2$:

$$\ln x^2 = 2 \ln x.$$

The logarithm of a number with an exponent | is | the product of the exponent and the logarithm of that number.

Figure 3.18 shows the graphs of $y = \ln x^2$ and $y = 2 \ln x$ in $[-5, 5, 1]$ by $[-5, 5, 1]$ viewing rectangles. Are $\ln x^2$ and $2 \ln x$ the same? The graphs illustrate that $y = \ln x^2$ and $y = 2 \ln x$ have different domains. The graphs are only the same if $x > 0$. Thus, we should write

$$\ln x^2 = 2 \ln x \text{ for } x > 0.$$

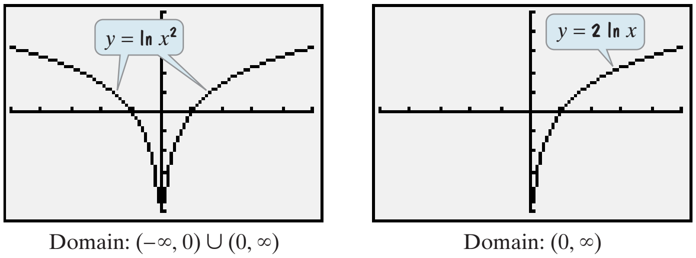


Figure 3.18 $\ln x^2$ and $2 \ln x$ have different domains.

When expanding a logarithmic expression, you might want to determine whether the rewriting has changed the domain of the expression. For the rest of this section, assume that all variables and variable expressions represent positive numbers.

EXAMPLE 3 Using the Power Rule

Use the power rule to expand each logarithmic expression:

a. $\log_5 7^4$ **b.** $\ln\sqrt{x}$ **c.** $\log(4x)^5$.

Solution

a. $\log_5 7^4 = 4 \log_5 7$ — The logarithm of a number with an exponent is the exponent times the logarithm of the number.

b. $\ln\sqrt{x} = \ln x^{\frac{1}{2}}$ — Rewrite the radical using a rational exponent.

$= \frac{1}{2}\ln x$ — Use the power rule to bring the exponent to the front.

c. $\log(4x)^5 = 5 \log(4x)$ — We immediately apply the power rule because the entire variable expression, 4x, is raised to the 5th power.

Check Point 3 Use the power rule to expand each logarithmic expression:

a. $\log_6 3^9$ **b.** $\ln\sqrt[3]{x}$ **c.** $\log(x + 4)^2$.

4 Expand logarithmic expressions.

Expanding Logarithmic Expressions

It is sometimes necessary to use more than one property of logarithms when you expand a logarithmic expression. Properties for expanding logarithmic expressions are as follows:

Properties for Expanding Logarithmic Expressions

For $M > 0$ and $N > 0$:

1. $\log_b(MN) = \log_b M + \log_b N$ — Product rule
2. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ — Quotient rule
3. $\log_b M^p = p \log_b M$ — Power rule

Study Tip

The graphs show that

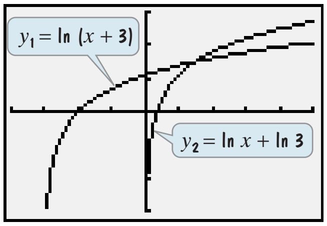
$$\ln(x+3) \neq \ln x + \ln 3.$$

$y = \ln x$ shifted 3 units left

$y = \ln x$ shifted $\ln 3$ units up

In general,

$$\log_b(M+N) \neq \log_b M + \log_b N.$$



[–4, 5, 1] by [–3, 3, 1]

Try to avoid the following errors:

Incorrect!

$$\log_b(M+N) = \log_b M + \log_b N$$
$$\log_b(M-N) = \log_b M - \log_b N$$
$$\log_b(M \cdot N) = \log_b M \cdot \log_b N$$
$$\log_b\left(\frac{M}{N}\right) = \frac{\log_b M}{\log_b N}$$
$$\frac{\log_b M}{\log_b N} = \log_b M - \log_b N$$
$$\log_b(MN^p) = p\log_b(MN)$$

EXAMPLE 4 Expanding Logarithmic Expressions

Use logarithmic properties to expand each expression as much as possible:

a. $\log_b(x^2\sqrt{y})$ **b.** $\log_6\left(\frac{\sqrt[3]{x}}{36y^4}\right)$.

Solution We will have to use two or more of the properties for expanding logarithms in each part of this example.

a.
$$\log_b(x^2\sqrt{y}) = \log_b\left(x^2y^{\frac{1}{2}}\right) \quad \text{Use exponential notation.}$$
$$= \log_b x^2 + \log_b y^{\frac{1}{2}} \quad \text{Use the product rule.}$$
$$= 2\log_b x + \frac{1}{2}\log_b y \quad \text{Use the power rule.}$$

b.
$$\log_6\left(\frac{\sqrt[3]{x}}{36y^4}\right) = \log_6\left(\frac{x^{\frac{1}{3}}}{36y^4}\right) \quad \text{Use exponential notation.}$$
$$= \log_6 x^{\frac{1}{3}} - \log_6(36y^4) \quad \text{Use the quotient rule.}$$
$$= \log_6 x^{\frac{1}{3}} - (\log_6 36 + \log_6 y^4) \quad \text{Use the product rule on } \log_6(36y^4).$$
$$= \frac{1}{3}\log_6 x - (\log_6 36 + 4\log_6 y) \quad \text{Use the power rule.}$$
$$= \frac{1}{3}\log_6 x - \log_6 36 - 4\log_6 y \quad \text{Apply the distributive property.}$$
$$= \frac{1}{3}\log_6 x - 2 - 4\log_6 y \quad \log_6 36 = 2 \text{ because 2 is the power to which we must raise 6 to get 36. } (6^2 = 36)$$

Check Point 4 Use logarithmic properties to expand each expression as much as possible:

a. $\log_b\left(x^4\sqrt[3]{y}\right)$ **b.** $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$.

5 Condense logarithmic expressions.

Condensing Logarithmic Expressions

To **condense a logarithmic expression**, we write the sum or difference of two or more logarithmic expressions as a single logarithmic expression. We use the properties of logarithms to do so.

Study Tip

These properties are the same as those in the box on page 416. The only difference is that we've reversed the sides in each property from the previous box.

Properties for Condensing Logarithmic Expressions

For $M > 0$ and $N > 0$:

1. $\log_b M + \log_b N = \log_b(MN)$ Product rule
2. $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$ Quotient rule
3. $p \log_b M = \log_b M^p$ Power rule

EXAMPLE 5 Condensing Logarithmic Expressions

Write as a single logarithm:

a. $\log_4 2 + \log_4 32$ **b.** $\log(4x - 3) - \log x$.

Solution

a. $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Use the product rule.

$= \log_4 64$ We now have a single logarithm. However, we can simplify.

$= 3$ $\log_4 64 = 3$ because $4^3 = 64$.

b. $\log(4x - 3) - \log x = \log\left(\frac{4x - 3}{x}\right)$ Use the quotient rule.

Check Point 5 Write as a single logarithm:

a. $\log 25 + \log 4$ **b.** $\log(7x + 6) - \log x$.

Coefficients of logarithms must be 1 before you can condense them using the product and quotient rules. For example, to condense

$$2 \ln x + \ln(x + 1),$$

the coefficient of the first term must be 1. We use the power rule to rewrite the coefficient as an exponent:

1. Use the power rule to make the number in front an exponent.

$$2 \ln x + \ln(x + 1) = \ln x^2 + \ln(x + 1) = \ln[x^2(x + 1)].$$

2. Use the product rule. The sum of logarithms with coefficients of 1 is the logarithm of the product.

EXAMPLE 6 Condensing Logarithmic Expressions

Write as a single logarithm:

a. $\frac{1}{2}\log x + 4\log(x - 1)$ **b.** $3\ln(x + 7) - \ln x$

c. $4\log_b x - 2\log_b 6 - \frac{1}{2}\log_b y$.

Solution

a. $\frac{1}{2}\log x + 4\log(x-1)$

$= \log x^{\frac{1}{2}} + \log(x-1)^4$ Use the power rule so that all coefficients are 1.

$= \log\left[x^{\frac{1}{2}}(x-1)^4\right]$ Use the product rule. The condensed form can be expressed as $\log[\sqrt{x}(x-1)^4]$.

b. $3\ln(x+7) - \ln x$

$= \ln(x+7)^3 - \ln x$ Use the power rule so that all coefficients are 1.

$= \ln\left[\frac{(x+7)^3}{x}\right]$ Use the quotient rule.

c. $4\log_b x - 2\log_b 6 - \frac{1}{2}\log_b y$

$= \log_b x^4 - \log_b 6^2 - \log_b y^{\frac{1}{2}}$ Use the power rule so that all coefficients are 1.

$= \log_b x^4 - \left(\log_b 36 + \log_b y^{\frac{1}{2}}\right)$ Rewrite as a single subtraction.

$= \log_b x^4 - \log_b\left(36y^{\frac{1}{2}}\right)$ Use the product rule.

$= \log_b\left(\frac{x^4}{36y^{\frac{1}{2}}}\right)$ or $\log_b\left(\frac{x^4}{36\sqrt{y}}\right)$ Use the quotient rule.

Check Point 6 Write as a single logarithm:

a. $2\ln x + \frac{1}{3}\ln(x+5)$ **b.** $2\log(x-3) - \log x$

c. $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$.

6 Use the change-of-base property.

The Change-of-Base Property

We have seen that calculators give the values of both common logarithms (base 10) and natural logarithms (base e). To find a logarithm with any other base, we can use the following change-of-base property:

The Change-of-Base Property

For any logarithmic bases a and b, and any positive number M,

$$\log_b M = \frac{\log_a M}{\log_a b}.$$

The logarithm of M with base b is equal to the logarithm of M with any new base divided by the logarithm of b with that new base.

In the change-of-base property, base b is the base of the original logarithm. Base a is a new base that we introduce. Thus, the change-of-base property allows us to change from base b to *any* new base a, as long as the newly introduced base is a positive number not equal to 1.

The change-of-base property is used to write a logarithm in terms of quantities that can be evaluated with a calculator. Because calculators contain keys for common (base 10) and natural (base e) logarithms, we will frequently introduce base 10 or base e.

Change-of-Base Property	Introducing Common Logarithms	Introducing Natural Logarithms
$\log_b M = \frac{\log_a M}{\log_a b}$ a is the new introduced base.	$\log_b M = \frac{\log_{10} M}{\log_{10} b}$ 10 is the new introduced base.	$\log_b M = \frac{\log_e M}{\log_e b}$ e is the new introduced base.

Using the notations for common logarithms and natural logarithms, we have the following results:

The Change-of-Base Property: Introducing Common and Natural Logarithms

Introducing Common Logarithms

$$\log_b M = \frac{\log M}{\log b}$$

Introducing Natural Logarithms

$$\log_b M = \frac{\ln M}{\ln b}$$

EXAMPLE 7 Changing Base to Common Logarithms

Use common logarithms to evaluate $\log_5 140$.

Solution Because $\log_b M = \frac{\log M}{\log b}$,

$$\log_5 140 = \frac{\log 140}{\log 5}$$

$$\approx 3.07.$$

Use a calculator: 140 LOG ÷ 5 LOG = or LOG 140 ÷ LOG 5 ENTER. On some calculators, parentheses are needed after 140 and 5.

This means that $\log_5 140 \approx 3.07$.

Discovery

Find a reasonable estimate of $\log_5 140$ to the nearest whole number. To what power can you raise 5 in order to get 140? Compare your estimate to the value obtained in Example 7.

Check Point 7 Use common logarithms to evaluate $\log_7 2506$.

EXAMPLE 8 Changing Base to Natural Logarithms

Use natural logarithms to evaluate $\log_5 140$.

Solution Because $\log_b M = \frac{\ln M}{\ln b}$,

$$\log_5 140 = \frac{\ln 140}{\ln 5}$$

$$\approx 3.07.$$

Use a calculator: 140 LN ÷ 5 LN = or LN 140 ÷ LN 5 ENTER. On some calculators, parentheses are needed after 140 and 5.

We have again shown that $\log_5 140 \approx 3.07$.

Check Point 8 Use natural logarithms to evaluate $\log_7 2506$.

Technology

We can use the change-of-base property to graph logarithmic functions with bases other than 10 or e on a graphing utility. For example, **Figure 3.19** shows the graphs of

$$y = \log_2 x \quad \text{and} \quad y = \log_{20} x$$

in a $[0, 10, 1]$ by $[-3, 3, 1]$ viewing rectangle. Because $\log_2 x = \frac{\ln x}{\ln 2}$ and $\log_{20} x = \frac{\ln x}{\ln 20}$, the functions are entered as

y_1 = LN x ÷ LN 2

and y_2 = LN x ÷ LN 20.

On some calculators, parentheses are needed after x, 2, and 20.

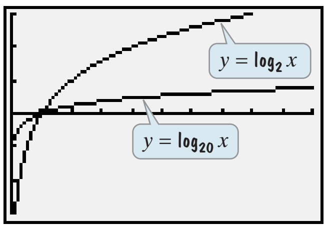


Figure 3.19 Using the change-of-base property to graph logarithmic functions

Exercise Set 3.3

Practice Exercises

In Exercises 1–40, use properties of logarithms to expand each logarithmic expression as much as possible. Where possible, evaluate logarithmic expressions without using a calculator.

1. $\log_5(7 \cdot 3)$
2. $\log_8(13 \cdot 7)$
3. $\log_7(7x)$
4. $\log_9(9x)$
5. $\log(1000x)$
6. $\log(10{,}000x)$
7. $\log_7\left(\frac{7}{x}\right)$
8. $\log_9\left(\frac{9}{x}\right)$
9. $\log\left(\frac{x}{100}\right)$
10. $\log\left(\frac{x}{1000}\right)$
11. $\log_4\left(\frac{64}{y}\right)$
12. $\log_5\left(\frac{125}{y}\right)$
13. $\ln\left(\frac{e^2}{5}\right)$
14. $\ln\left(\frac{e^4}{8}\right)$
15. $\log_b x^3$
16. $\log_b x^7$
17. $\log N^{-6}$
18. $\log M^{-8}$
19. $\ln\sqrt[5]{x}$
20. $\ln\sqrt[7]{x}$
21. $\log_b(x^2y)$
22. $\log_b(xy^3)$
23. $\log_4\left(\frac{\sqrt{x}}{64}\right)$
24. $\log_5\left(\frac{\sqrt{x}}{25}\right)$
25. $\log_6\left(\frac{36}{\sqrt{x+1}}\right)$
26. $\log_8\left(\frac{64}{\sqrt{x+1}}\right)$
27. $\log_b\left(\frac{x^2y}{z^2}\right)$
28. $\log_b\left(\frac{x^3y}{z^2}\right)$
29. $\log\sqrt{100x}$
30. $\ln\sqrt{ex}$
31. $\log\sqrt[3]{\frac{x}{y}}$
32. $\log\sqrt[5]{\frac{x}{y}}$
33. $\log_b\left(\frac{\sqrt{xy^3}}{z^3}\right)$
34. $\log_b\left(\frac{\sqrt[3]{xy^4}}{z^5}\right)$
35. $\log_5\sqrt[3]{\frac{x^2y}{25}}$
36. $\log_2\sqrt[5]{\frac{xy^4}{16}}$
37. $\ln\left[\frac{x^3\sqrt{x^2+1}}{(x+1)^4}\right]$
38. $\ln\left[\frac{x^4\sqrt{x^2+3}}{(x+3)^5}\right]$
39. $\log\left[\frac{10x^2\sqrt[3]{1-x}}{7(x+1)^2}\right]$
40. $\log\left[\frac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right]$

In Exercises 41–70, use properties of logarithms to condense each logarithmic expression. Write the expression as a single logarithm whose coefficient is 1. Where possible, evaluate logarithmic expressions without using a calculator.

41. $\log 5 + \log 2$
42. $\log 250 + \log 4$
43. $\ln x + \ln 7$
44. $\ln x + \ln 3$
45. $\log_2 96 - \log_2 3$
46. $\log_3 405 - \log_3 5$
47. $\log(2x+5) - \log x$
48. $\log(3x+7) - \log x$
49. $\log x + 3\log y$
50. $\log x + 7\log y$
51. $\frac{1}{2}\ln x + \ln y$
52. $\frac{1}{3}\ln x + \ln y$
53. $2\log_b x + 3\log_b y$
54. $5\log_b x + 6\log_b y$
55. $5\ln x - 2\ln y$
56. $7\ln x - 3\ln y$
57. $3\ln x - \frac{1}{3}\ln y$
58. $2\ln x - \frac{1}{2}\ln y$
59. $4\ln(x+6) - 3\ln x$
60. $8\ln(x+9) - 4\ln x$
61. $3\ln x + 5\ln y - 6\ln z$
62. $4\ln x + 7\ln y - 3\ln z$
63. $\frac{1}{2}(\log x + \log y)$
64. $\frac{1}{3}(\log_4 x - \log_4 y)$
65. $\frac{1}{2}(\log_5 x + \log_5 y) - 2\log_5(x+1)$
66. $\frac{1}{3}(\log_4 x - \log_4 y) + 2\log_4(x+1)$
67. $\frac{1}{3}[2\ln(x+5) - \ln x - \ln(x^2-4)]$
68. $\frac{1}{3}[5\ln(x+6) - \ln x - \ln(x^2-25)]$
69. $\log x + \log(x^2-1) - \log 7 - \log(x+1)$
70. $\log x + \log(x^2-4) - \log 15 - \log(x+2)$

In Exercises 71–78, use common logarithms or natural logarithms and a calculator to evaluate to four decimal places.

71. $\log_5 13$
72. $\log_6 17$
73. $\log_{14} 87.5$
74. $\log_{16} 57.2$
75. $\log_{0.1} 17$
76. $\log_{0.3} 19$
77. $\log_\pi 63$
78. $\log_\pi 400$

In Exercises 79–82, use a graphing utility and the change-of-base property to graph each function.

79. $y = \log_3 x$
80. $y = \log_{15} x$
81. $y = \log_2(x+2)$
82. $y = \log_3(x-2)$

Practice Plus

In Exercises 83–88, let $\log_b 2 = A$ and $\log_b 3 = C$. Write each expression in terms of A and C.

83. $\log_b \frac{3}{2}$
84. $\log_b 6$
85. $\log_b 8$
86. $\log_b 81$
87. $\log_b\sqrt{\frac{2}{27}}$
88. $\log_b\sqrt{\frac{3}{16}}$

In Exercises 89–102, determine whether each equation is true or false. Where possible, show work to support your conclusion. If the statement is false, make the necessary change(s) to produce a true statement.

89. $\ln e = 0$
90. $\ln 0 = e$
91. $\log_4(2x^3) = 3\log_4(2x)$
92. $\ln(8x^3) = 3\ln(2x)$
93. $x\log 10^x = x^2$
94. $\ln(x+1) = \ln x + \ln 1$
95. $\ln(5x) + \ln 1 = \ln(5x)$
96. $\ln x + \ln(2x) = \ln(3x)$
97. $\log(x+3) - \log(2x) = \dfrac{\log(x+3)}{\log(2x)}$
98. $\dfrac{\log(x+2)}{\log(x-1)} = \log(x+2) - \log(x-1)$
99. $\log_6\left(\frac{x-1}{x^2+4}\right) = \log_6(x-1) - \log_6(x^2+4)$
100. $\log_6[4(x+1)] = \log_6 4 + \log_6(x+1)$
101. $\log_3 7 = \dfrac{1}{\log_7 3}$
102. $e^x = \dfrac{1}{\ln x}$

Application Exercises

103. The loudness level of a sound can be expressed by comparing the sound's intensity to the intensity of a sound barely audible to the human ear. The formula

$$D = 10(\log I - \log I_0)$$

describes the loudness level of a sound, D, in decibels, where I is the intensity of the sound, in watts per meter^2, and I_0 is the intensity of a sound barely audible to the human ear.

a. Express the formula so that the expression in parentheses is written as a single logarithm.

b. Use the form of the formula from part (a) to answer this question: If a sound has an intensity 100 times the intensity of a softer sound, how much larger on the decibel scale is the loudness level of the more intense sound?

104. The formula

$$t = \frac{1}{c}[\ln A - \ln(A - N)]$$

describes the time, t, in weeks, that it takes to achieve mastery of a portion of a task, where A is the maximum learning possible, N is the portion of the learning that is to be achieved, and c is a constant used to measure an individual's learning style.

a. Express the formula so that the expression in brackets is written as a single logarithm.

b. The formula is also used to determine how long it will take chimpanzees and apes to master a task. For example, a typical chimpanzee learning sign language can master a maximum of 65 signs. Use the form of the formula from part (a) to answer this question: How many weeks will it take a chimpanzee to master 30 signs if c for that chimp is 0.03?

Writing in Mathematics

105. Describe the product rule for logarithms and give an example.

106. Describe the quotient rule for logarithms and give an example.

107. Describe the power rule for logarithms and give an example.

108. Without showing the details, explain how to condense $\ln x - 2\ln(x+1)$.

109. Describe the change-of-base property and give an example.

110. Explain how to use your calculator to find $\log_{14} 283$.

111. You overhear a student talking about a property of logarithms in which division becomes subtraction. Explain what the student means by this.

112. Find ln 2 using a calculator. Then calculate each of the following: $1 - \frac{1}{2}$; $1 - \frac{1}{2} + \frac{1}{3}$; $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$; $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$; .... Describe what you observe.

Technology Exercises

113. a. Use a graphing utility (and the change-of-base property) to graph $y = \log_3 x$.

b. Graph $y = 2 + \log_3 x$, $y = \log_3(x+2)$, and $y = -\log_3 x$ in the same viewing rectangle as $y = \log_3 x$. Then describe the change or changes that need to be made to the graph of $y = \log_3 x$ to obtain each of these three graphs.

114. Graph $y = \log x$, $y = \log(10x)$, and $y = \log(0.1x)$ in the same viewing rectangle. Describe the relationship among the three graphs. What logarithmic property accounts for this relationship?

115. Use a graphing utility and the change-of-base property to graph $y = \log_3 x$, $y = \log_{25} x$, and $y = \log_{100} x$ in the same viewing rectangle.

a. Which graph is on the top in the interval $(0, 1)$? Which is on the bottom?

b. Which graph is on the top in the interval $(1, \infty)$? Which is on the bottom?

c. Generalize by writing a statement about which graph is on top, which is on the bottom, and in which intervals, using $y = \log_b x$ where $b > 1$.

Disprove each statement in Exercises 116–120 by

a. *letting y equal a positive constant of your choice, and*

b. *using a graphing utility to graph the function on each side of the equal sign. The two functions should have different graphs, showing that the equation is not true in general.*

116. $\log(x + y) = \log x + \log y$

117. $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

118. $\ln(x - y) = \ln x - \ln y$

119. $\ln(xy) = (\ln x)(\ln y)$

120. $\frac{\ln x}{\ln y} = \ln x - \ln y$

Critical Thinking Exercises

Make Sense? *In Exercises 121–124, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

121. Because I cannot simplify the expression $b^m + b^n$ by adding exponents, there is no property for the logarithm of a sum.

122. Because logarithms are exponents, the product, quotient, and power rules remind me of properties for operations with exponents.

123. I can use any positive number other than 1 in the change-of-base property, but the only practical bases are 10 and e because my calculator gives logarithms for these two bases.

124. I expanded $\log_4 \sqrt{\frac{x}{y}}$ by writing the radical using a rational exponent and then applying the quotient rule, obtaining $\frac{1}{2}\log_4 x - \log_4 y$.

In Exercises 125–132, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

125. $\ln\sqrt{2} = \frac{\ln 2}{2}$

126. $\frac{\log_7 49}{\log_7 7} = \log_7 49 - \log_7 7$

127. $\log_b(x^3 + y^3) = 3\log_b x + 3\log_b y$

128. $\log_b(xy)^5 = (\log_b x + \log_b y)^5$

129. Use the change-of-base property to prove that

$$\log e = \frac{1}{\ln 10}.$$

130. If $\log 3 = A$ and $\log 7 = B$, find $\log_7 9$ in terms of A and B.

131. Write as a single term that does not contain a logarithm:

$$e^{\ln 8x^5 - \ln 2x^2}.$$

132. If $f(x) = \log_b x$, show that

$$\frac{f(x+h) - f(x)}{h} = \log_b\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}, \ h \neq 0.$$

Preview Exercises

Exercises 133–135 will help you prepare for the material covered in the next section.

133. Solve for x: $a(x - 2) = b(2x + 3)$.

134. Solve: $x(x - 7) = 3$.

135. Solve: $\frac{x+2}{4x+3} = \frac{1}{x}$.