73. Standing under this arch, I can determine its height by measuring the angle of elevation to the top of the arch and my distance to a point directly under the arch.

Delicate Arch in Arches National Park, Utah

In Exercises 74–77, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

74. $\dfrac{\tan 45°}{\tan 15°} = \tan 3°$

75. $\tan^2 15° - \sec^2 15° = -1$

76. $\sin 45° + \cos 45° = 1$

77. $\tan^2 5° = \tan 25°$

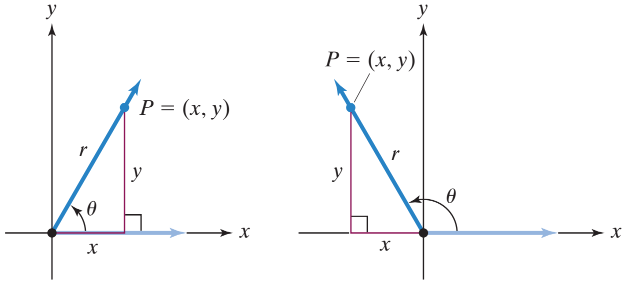
78. Explain why the sine or cosine of an acute angle cannot be greater than or equal to 1.

79. Describe what happens to the tangent of an acute angle as the angle gets close to 90°. What happens at 90°?

80. From the top of a 250-foot lighthouse, a plane is sighted overhead and a ship is observed directly below the plane. The angle of elevation of the plane is 22° and the angle of depression of the ship is 35°. Find **a.** the distance of the ship from the lighthouse; **b.** the plane's height above the water. Round to the nearest foot.

Preview Exercises

Exercises 81–83 will help you prepare for the material covered in the next section. Use these figures to solve Exercises 81–82.



(a) θ lies in quadrant I. **(b)** θ lies in quadrant II.

81. a. Write a ratio that expresses $\sin\theta$ for the right triangle in **Figure (a)**.

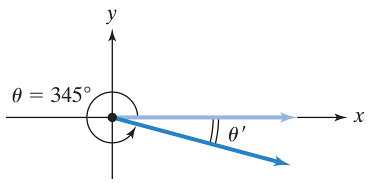
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 4$. Is this ratio positive or negative?

82. a. Write a ratio that expresses $\cos\theta$ for the right triangle in **Figure (a)**.

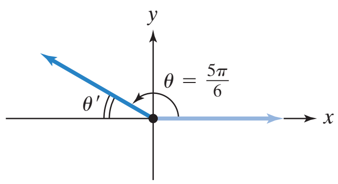
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 5$. Is this ratio positive or negative?

83. Find the positive angle θ' formed by the terminal side of θ and the x-axis.

a.



b.



Section 4.4 Trigonometric Functions of Any Angle

Objectives

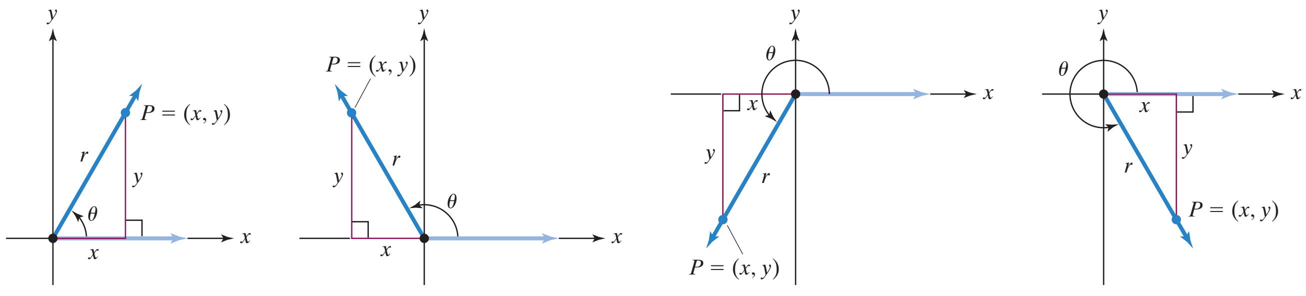
1. Use the definitions of trigonometric functions of any angle.
2. Use the signs of the trigonometric functions.
3. Find reference angles.
4. Use reference angles to evaluate trigonometric functions.

Cycles govern many aspects of life—heartbeats, sleep patterns, seasons, and tides all follow regular, predictable cycles. Because of their periodic nature, trigonometric functions are used to model phenomena that occur in cycles. It is helpful to apply these models regardless of whether we think of the domains of trigonometentric functions as sets of real numbers or sets of angles. In order to understand and use models for cyclic phenomena from an angle perspective, we need to move beyond right triangles.

1 Use the definitions of trigonometric functions of any angle.

Trigonometric Functions of Any Angle

In the last section, we evaluated trigonometric functions of acute angles, such as that shown in **Figure 4.41(a)**. Note that this angle is in standard position. The point $P = (x, y)$ is a point r units from the origin on the terminal side of θ. A right triangle is formed by drawing a line segment from $P = (x, y)$ perpendicular to the x-axis. Note that y is the length of the side opposite θ and x is the length of the side adjacent to θ.



(a) θ lies in quadrant I. **(b)** θ lies in quadrant II. **(c)** θ lies in quadrant III. **(d)** θ lies in quadrant IV.

Figure 4.41

Figures 4.41(b), **(c)**, and **(d)** show angles in standard position, but they are not acute. We can extend our definitions of the six trigonometric functions to include such angles, as well as quadrantal angles. (Recall that a quadrantal angle has its terminal side on the x-axis or y-axis; such angles are *not* shown in **Figure 4.41**.) The point $P = (x, y)$ may be any point on the terminal side of the angle θ other than the origin, (0, 0).

Study Tip

If θ is acute, we have the right triangle shown in **Figure 4.41(a)**. In this situation, the definitions in the box are the right triangle definitions of the trigonometric functions. This should make it easier for you to remember the six definitions.

Definitions of Trigonometric Functions of Any Angle

Let θ be any angle in standard position and let $P = (x, y)$ be a point on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from (0, 0) to (x, y), as shown in **Figure 4.41**, the **six trigonometric functions of θ** are defined by the following ratios:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0.$$

The ratios in the second column are the reciprocals of the corresponding ratios in the first column.

Because the point $P = (x, y)$ is any point on the terminal side of θ other than the origin, (0, 0), $r = \sqrt{x^2 + y^2}$ cannot be zero. Examine the six trigonometric functions defined above. Note that the denominator of the sine and cosine functions is r. Because $r \neq 0$, the sine and cosine functions are defined for any angle θ. This is not true for the other four trigonometric functions. Note that the denominator of the tangent and secant functions is x: $\tan\theta = \frac{y}{x}$ and $\sec\theta = \frac{r}{x}$. These functions are not defined if $x = 0$. If the point $P = (x, y)$ is on the y-axis, then $x = 0$. Thus, the tangent and secant functions are undefined for all quadrantal angles with terminal sides on the positive or negative y-axis. Likewise, if $P = (x, y)$ is on the x-axis, then $y = 0$, and the cotangent and cosecant functions are undefined: $\cot\theta = \frac{x}{y}$ and $\csc\theta = \frac{r}{y}$. The cotangent and cosecant functions are undefined for all quadrantal angles with terminal sides on the positive or negative x-axis.

EXAMPLE 1 Evaluating Trigonometric Functions

Let $P = (-3, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

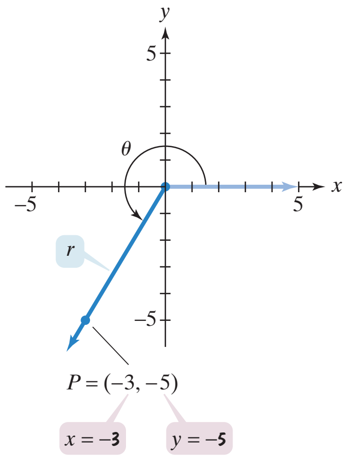


Figure 4.42

Solution The situation is shown in **Figure 4.42**. We need values for x, y, and r to evaluate all six trigonometric functions. We are given the values of x and y. Because $P = (-3, -5)$ is a point on the terminal side of θ, $x = -3$ and $y = -5$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}.$$

Now that we know x, y, and r, we can find the six trigonometric functions of θ. Where appropriate, we will rationalize denominators.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{34}} = -\frac{5}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{5\sqrt{34}}{34} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{-5} = -\frac{\sqrt{34}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-3} = \frac{5}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-5} = \frac{3}{5}$$

Check Point 1 Let $P = (1, -3)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

How do we find the values of the trigonometric functions for a quadrantal angle? First, draw the angle in standard position. Second, choose a point P on the angle's terminal side. The trigonometric function values of θ depend only on the size of θ and not on the distance of point P from the origin. Thus, we will choose a point that is 1 unit from the origin. Finally, apply the definitions of the appropriate trigonometric functions.

EXAMPLE 2 Trigonometric Functions of Quadrantal Angles

Evaluate, if possible, the sine function and the tangent function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

Solution

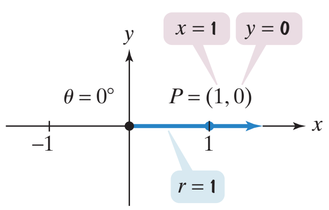


Figure 4.43

a. If $\theta = 0° = 0$ radians, then the terminal side of the angle is on the positive x-axis. Let us select the point $P = (1, 0)$ with $x = 1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.43** shows values of x, y, and r corresponding to $\theta = 0°$ or 0 radians. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 0° = \sin 0 = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 0° = \tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

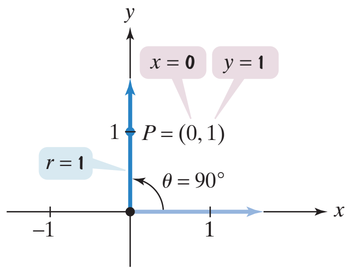


Figure 4.44

b. If $\theta = 90° = \frac{\pi}{2}$ radians, then the terminal side of the angle is on the positive y-axis. Let us select the point $P = (0, 1)$ with $x = 0$ and $y = 1$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.44** shows values of x, y, and r corresponding to $\theta = 90°$ or $\frac{\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 90° = \sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$$

$$\tan 90° = \tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0}$$

Because division by 0 is undefined, tan 90° is undefined.

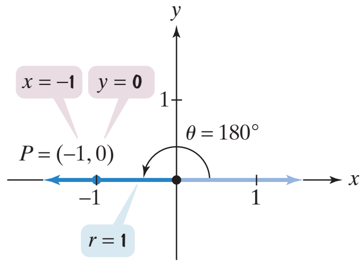


Figure 4.45

c. If $\theta = 180° = \pi$ radians, then the terminal side of the angle is on the negative x-axis. Let us select the point $P = (-1, 0)$ with $x = -1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.45** shows values of x, y, and r corresponding to $\theta = 180°$ or π. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 180° = \sin\pi = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 180° = \tan\pi = \frac{y}{x} = \frac{0}{-1} = 0$$

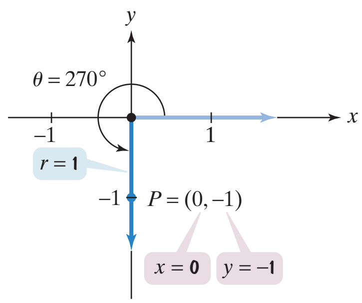


Figure 4.46

d. If $\theta = 270° = \frac{3\pi}{2}$ radians, then the terminal side of the angle is on the negative y-axis. Let us select the point $P = (0, -1)$ with $x = 0$ and $y = -1$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.46** shows values of x, y, and r corresponding to $\theta = 270°$ or $\frac{3\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 270° = \sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\tan 270° = \tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0}$$

Because division by 0 is undefined, tan 270° is undefined.

Discovery

Try finding tan 90° and tan 270° with your calculator. Describe what occurs.

Check Point 2 Evaluate, if possible, the cosine function and the cosecant function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

2 Use the signs of the trigonometric functions.

The Signs of the Trigonometric Functions

In Example 2, we evaluated trigonometric functions of quadrantal angles. However, we will now return to the trigonometric functions of nonquadrantal angles. **If θ is not a quadrantal angle, the sign of a trigonometric function depends on the quadrant in which θ lies.** In all four quadrants, r is positive. However, x and y can be positive or negative. For example, if θ lies in quadrant II, x is negative and y is positive. Thus, the only positive ratios in this quadrant are $\frac{y}{r}$ and its reciprocal, $\frac{r}{y}$. These ratios are the function values for the sine and cosecant, respectively. In short, if θ lies in quadrant II, $\sin\theta$ and $\csc\theta$ are positive. The other four trigonometric functions are negative.

Figure 4.47 summarizes the signs of the trigonometric functions. If θ lies in quadrant I, all six functions are positive. If θ lies in quadrant II, only $\sin\theta$ and $\csc\theta$ are positive. If θ lies in quadrant III, only $\tan\theta$ and $\cot\theta$ are positive. Finally, if θ lies in quadrant IV, only $\cos\theta$ and $\sec\theta$ are positive. Observe that the positive functions in each quadrant occur in reciprocal pairs.

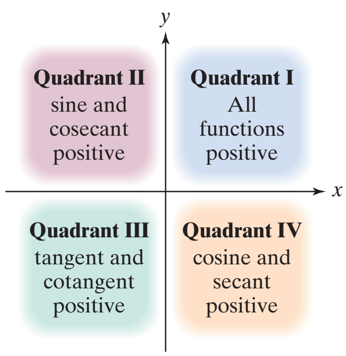


Figure 4.47 The signs of the trigonometric functions

Study Tip

Here's a phrase to help you remember the signs of the trig functions:

A	Smart	Trig	Class.
All trig functions are positive in **QI.**	Sine and its reciprocal, cosecant, are positive in **QII.**	Tangent and its reciprocal, cotangent, are positive in **QIII.**	Cosine and its reciprocal, secant, are positive in **QIV.**

EXAMPLE 3 Finding the Quadrant in Which an Angle Lies

If $\tan\theta < 0$ and $\cos\theta > 0$, name the quadrant in which angle θ lies.

Solution When $\tan\theta < 0$, θ lies in quadrant II or IV. When $\cos\theta > 0$, θ lies in quadrant I or IV. When both conditions are met ($\tan\theta < 0$ and $\cos\theta > 0$), θ must lie in quadrant IV.

Check Point 3 If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which angle θ lies.

EXAMPLE 4 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$, find $\cos\theta$ and $\csc\theta$.

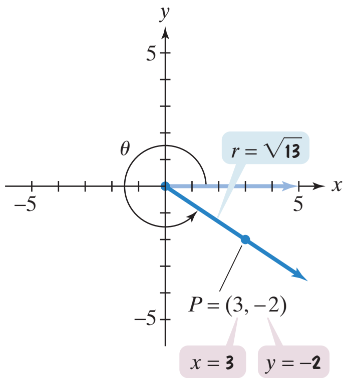


Figure 4.48 $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$

Solution Because the tangent is negative and the cosine is positive, θ lies in quadrant IV. This will help us to determine whether the negative sign in $\tan\theta = -\frac{2}{3}$ should be associated with the numerator or the denominator. Keep in mind that in quadrant IV, x is positive and y is negative. Thus,

In quadrant IV, y is negative.

$$\tan\theta = -\frac{2}{3} = \frac{y}{x} = \frac{-2}{3}.$$

(See **Figure 4.48**.) Thus, $x = 3$ and $y = -2$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}.$$

Now that we know x, y, and r, we can find $\cos\theta$ and $\csc\theta$.

$$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{-2} = -\frac{\sqrt{13}}{2}$$

Check Point 4 Given $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.

In Example 4, we used the quadrant in which θ lies to determine whether a negative sign should be associated with the numerator or the denominator. Here's a situation, similar to Example 4, where negative signs should be associated with *both* the numerator and the denominator:

$$\tan\theta = \frac{3}{5} \quad \text{and} \quad \cos\theta < 0.$$

Because the tangent is positive and the cosine is negative, θ lies in quadrant III. In quadrant III, x is negative and y is negative. Thus,

$$\tan\theta = \frac{3}{5} = \frac{y}{x} = \frac{-3}{-5}.$$

We see that $x = -5$ and $y = -3$.

3 Find reference angles.

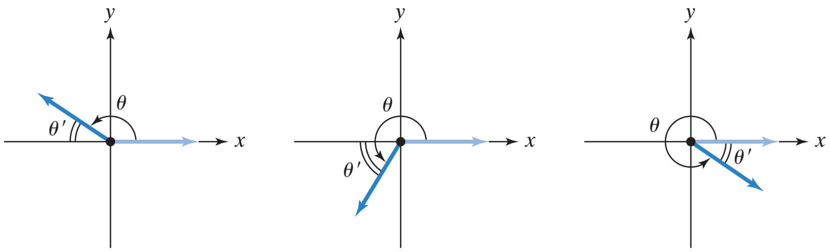
Reference Angles

We will often evaluate trigonometric functions of positive angles greater than 90° and all negative angles by making use of a positive acute angle. This positive acute angle is called a *reference angle.*

Definition of a Reference Angle

Let θ be a nonacute angle in standard position that lies in a quadrant. Its **reference angle** is the positive acute angle θ' formed by the terminal side of θ and the x-axis.

Figure 4.49 shows the reference angle for θ lying in quadrants II, III, and IV. Notice that the formula used to find θ', the reference angle, varies according to the quadrant in which θ lies. You may find it easier to find the reference angle for a given angle by making a figure that shows the angle in standard position. The acute angle formed by the terminal side of this angle and the x-axis is the reference angle.



If $90° < \theta < 180°$, then $\theta' = 180° - \theta$.

If $180° < \theta < 270°$, then $\theta' = \theta - 180°$.

If $270° < \theta < 360°$, then $\theta' = 360° - \theta$.

Figure 4.49 Reference angles, θ', for positive angles, θ, in quadrants II, III, and IV

EXAMPLE 5 Finding Reference Angles

Find the reference angle, θ', for each of the following angles:

a. $\theta = 345°$ **b.** $\theta = \frac{5\pi}{6}$ **c.** $\theta = -135°$ **d.** $\theta = 2.5$.

Solution

a. A 345° angle in standard position is shown in **Figure 4.50**. Because 345° lies in quadrant IV, the reference angle is

$$\theta' = 360° - 345° = 15°.$$

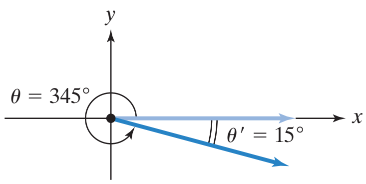


Figure 4.50

b. Because $\frac{5\pi}{6}$ lies between $\frac{\pi}{2} = \frac{3\pi}{6}$ and $\pi = \frac{6\pi}{6}$, $\theta = \frac{5\pi}{6}$ lies in quadrant II. The angle is shown in **Figure 4.51**. The reference angle is

$$\theta' = \pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}.$$

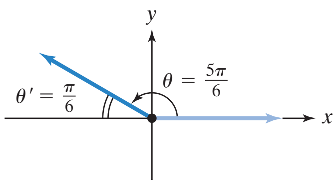


Figure 4.51

c. A $-135°$ angle in standard position is shown in **Figure 4.52**. The figure indicates that the positive acute angle formed by the terminal side of θ and the x-axis is 45°. The reference angle is

$$\theta' = 45°.$$

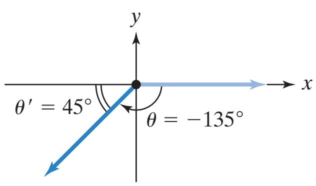


Figure 4.52

Discovery

Solve part (c) by first finding a positive coterminal angle for $-135°$ less than 360°. Use the positive coterminal angle to find the reference angle.

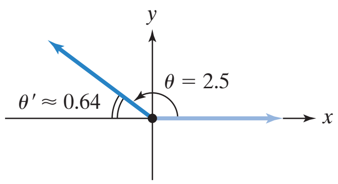


Figure 4.53

d. The angle $\theta = 2.5$ lies between $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$. This means that $\theta = 2.5$ is in quadrant II, shown in **Figure 4.53**. The reference angle is

$$\theta' = \pi - 2.5 \approx 0.64.$$

Check Point 5 Find the reference angle, θ', for each of the following angles:

a. $\theta = 210°$ **b.** $\theta = \frac{7\pi}{4}$ **c.** $\theta = -240°$ **d.** $\theta = 3.6.$

Finding reference angles for angles that are greater than $360° \,(2\pi)$ or less than $-360° \,(-2\pi)$ involves using coterminal angles. We have seen that coterminal angles have the same initial and terminal sides. Recall that coterminal angles can be obtained by increasing or decreasing an angle's measure by an integer multiple of $360°$ or 2π.

Finding Reference Angles for Angles Greater Than 360° (2π) or Less Than −360° (−2π)

1. Find a positive angle α less than $360°$ or 2π that is coterminal with the given angle.
2. Draw α in standard position.
3. Use the drawing to find the reference angle for the given angle. The positive acute angle formed by the terminal side of α and the x-axis is the reference angle.

EXAMPLE 6 Finding Reference Angles

Find the reference angle for each of the following angles:

a. $\theta = 580°$ **b.** $\theta = \frac{8\pi}{3}$ **c.** $\theta = -\frac{13\pi}{6}.$

Solution

a. For a $580°$ angle, subtract $360°$ to find a positive coterminal angle less than $360°$.

$$580° - 360° = 220°$$

Figure 4.54 shows $\alpha = 220°$ in standard position. Because $220°$ lies in quadrant III, the reference angle is

$$\alpha' = 220° - 180° = 40°.$$

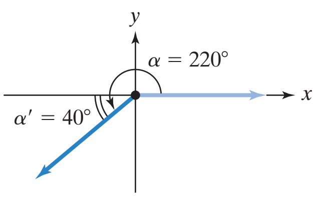


Figure 4.54

b. For an $\frac{8\pi}{3}$, or $2\frac{2}{3}\pi$, angle, subtract 2π to find a positive coterminal angle less than 2π.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

Figure 4.55 shows $\alpha = \frac{2\pi}{3}$ in standard position. Because $\frac{2\pi}{3}$ lies in quadrant II, the reference angle is

$$\alpha' = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}.$$

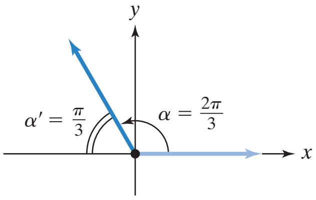


Figure 4.55

Discovery

Solve part (c) using the coterminal angle formed by adding 2π, rather than 4π, to the given angle.

c. For a $-\frac{13\pi}{6}$, or $-2\frac{1}{6}\pi$, angle, add 4π to find a positive coterminal angle less than 2π.

$$-\frac{13\pi}{6} + 4\pi = -\frac{13\pi}{6} + \frac{24\pi}{6} = \frac{11\pi}{6}$$

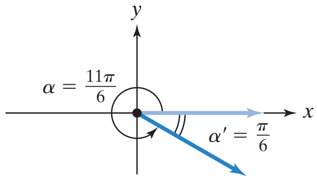


Figure 4.56

Figure 4.56 shows $\alpha = \frac{11\pi}{6}$ in standard position.

Because $\frac{11\pi}{6}$ lies in quadrant IV, the reference angle is

$$\alpha' = 2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}.$$

Check Point 6 Find the reference angle for each of the following angles:

a. $\theta = 665°$ **b.** $\theta = \frac{15\pi}{4}$ **c.** $\theta = -\frac{11\pi}{3}$.

4 Use reference angles to evaluate trigonometric functions.

Evaluating Trigonometric Functions Using Reference Angles

The way that reference angles are defined makes them useful in evaluating trigonometric functions.

Using Reference Angles to Evaluate Trigonometric Functions

The values of the trigonometric functions of a given angle, θ, are the same as the values of the trigonometric functions of the reference angle, θ', except possibly for the sign. A function value of the acute reference angle, θ', is always positive. However, the same function value for θ may be positive or negative.

For example, we can use a reference angle, θ', to obtain an exact value for $\tan 120°$. The reference angle for $\theta = 120°$ is $\theta' = 180° - 120° = 60°$. We know the exact value of the tangent function of the reference angle: $\tan 60° = \sqrt{3}$. We also know that the value of a trigonometric function of a given angle, θ, is the same as that of its reference angle, θ', except possibly for the sign. Thus, we can conclude that $\tan 120°$ equals $-\sqrt{3}$ or $\sqrt{3}$.

What sign should we attach to $\sqrt{3}$? A $120°$ angle lies in quadrant II, where only the sine and cosecant are positive. Thus, the tangent function is negative for a $120°$ angle. Therefore,

Prefix by a negative sign to show tangent is negative in quadrant II.

$$\tan 120° = -\tan 60° = -\sqrt{3}.$$

The reference angle for 120° is 60°.

In the previous section, we used two right triangles to find exact trigonometric values of $30°$, $45°$, and $60°$. Using a procedure similar to finding $\tan 120°$, we can now find the exact function values of all angles for which $30°$, $45°$, or $60°$ are reference angles.

A Procedure for Using Reference Angles to Evaluate Trigonometric Functions

The value of a trigonometric function of any angle θ is found as follows:

1. Find the associated reference angle, θ', and the function value for θ'.
2. Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1.

Discovery

Draw the two right triangles involving 30°, 45°, and 60°. Indicate the length of each side. Use these lengths to verify the function values for the reference angles in the solution to Example 7.

EXAMPLE 7 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\sin 135°$ **b.** $\cos \frac{4\pi}{3}$ **c.** $\cot\left(-\frac{\pi}{3}\right)$.

Solution

a. We use our two-step procedure to find $\sin 135°$.

Step 1 Find the reference angle, θ', and $\sin \theta'$. Figure 4.57 shows 135° lies in quadrant II. The reference angle is

$$\theta' = 180° - 135° = 45°.$$

The function value for the reference angle is $\sin 45° = \frac{\sqrt{2}}{2}$.

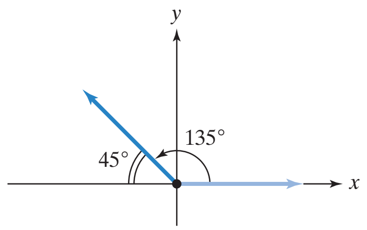


Figure 4.57 Reference angle for 135°

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = 135°$ lies in quadrant II. Because the sine is positive in quadrant II, we put a + sign before the function value of the reference angle. Thus,

The sine is positive in quadrant II.

$$\sin 135° = +\sin 45° = \frac{\sqrt{2}}{2}.$$

The reference angle for 135° is 45°.

b. We use our two-step procedure to find $\cos \frac{4\pi}{3}$.

Step 1 Find the reference angle, θ', and $\cos \theta'$. Figure 4.58 shows that $\theta = \frac{4\pi}{3}$ lies in quadrant III. The reference angle is

$$\theta' = \frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}.$$

The function value for the reference angle is

$$\cos \frac{\pi}{3} = \frac{1}{2}.$$

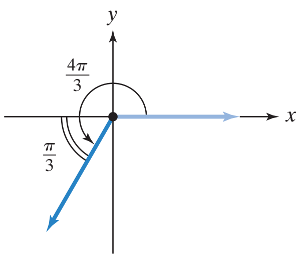


Figure 4.58 Reference angle for $\frac{4\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = \frac{4\pi}{3}$ lies in quadrant III. Because only the tangent and cotangent are positive in quadrant III, the cosine is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cosine is negative in quadrant III.

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}.$$

The reference angle for $\frac{4\pi}{3}$ is $\frac{\pi}{3}$.

c. We use our two-step procedure to find $\cot\left(-\frac{\pi}{3}\right)$.

Step 1 Find the reference angle, θ', and $\cot\theta'$. Figure 4.59 shows that $\theta = -\frac{\pi}{3}$ lies in quadrant IV. The reference angle is $\theta' = \frac{\pi}{3}$. The function value for the reference angle is $\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$.

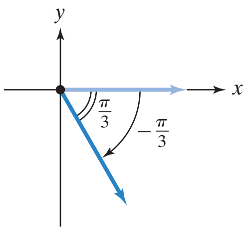


Figure 4.59 Reference angle for $-\frac{\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = -\frac{\pi}{3}$ lies in quadrant IV. Because only the cosine and secant are positive in quadrant IV, the cotangent is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cotangent is negative in quadrant IV.

$$\cot\left(-\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{\sqrt{3}}{3}.$$

The reference angle for $-\frac{\pi}{3}$ is $\frac{\pi}{3}$.

Check Point 7 Use reference angles to find the exact value of the following trigonometric functions:

a. $\sin 300°$ **b.** $\tan\frac{5\pi}{4}$ **c.** $\sec\left(-\frac{\pi}{6}\right)$.

In our final example, we use positive coterminal angles less than 2π to find the reference angles.

EXAMPLE 8 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\tan\frac{14\pi}{3}$ **b.** $\sec\left(-\frac{17\pi}{4}\right)$.

Solution

a. We use our two-step procedure to find $\tan\frac{14\pi}{3}$.

Step 1 Find the reference angle, θ', and $\tan\theta'$. Because the given angle, $\frac{14\pi}{3}$ or $4\frac{2}{3}\pi$, exceeds 2π, subtract 4π to find a positive coterminal angle less than 2π.

$$\theta = \frac{14\pi}{3} - 4\pi = \frac{14\pi}{3} - \frac{12\pi}{3} = \frac{2\pi}{3}$$

Figure 4.60 shows $\theta = \frac{2\pi}{3}$ in standard position. The angle lies in quadrant II. The reference angle is

$$\theta' = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}.$$

The function value for the reference angle is $\tan\frac{\pi}{3} = \sqrt{3}$.

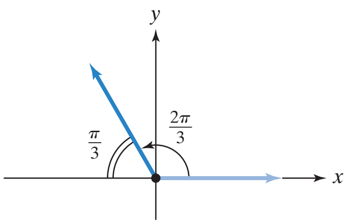


Figure 4.60 Reference angle for $\frac{2\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{2\pi}{3}$ lies in quadrant II. Because the tangent is negative in quadrant II, we put a − sign before the function value of the reference angle. Thus,

$$\tan\frac{14\pi}{3} = \tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

The tangent is negative in quadrant II.

The reference angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$.

b. We use our two-step procedure to find $\sec\left(-\frac{17\pi}{4}\right)$.

Step 1 Find the reference angle, θ', and $\sec\theta'$. Because the given angle, $-\frac{17\pi}{4}$ or $-4\frac{1}{4}\pi$, is less than -2π, add 6π (three multiples of 2π) to find a positive coterminal angle less than 2π.

$$\theta = -\frac{17\pi}{4} + 6\pi = -\frac{17\pi}{4} + \frac{24\pi}{4} = \frac{7\pi}{4}$$

Figure 4.61 shows $\theta = \frac{7\pi}{4}$ in standard position. The angle lies in quadrant IV. The reference angle is

$$\theta' = 2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}.$$

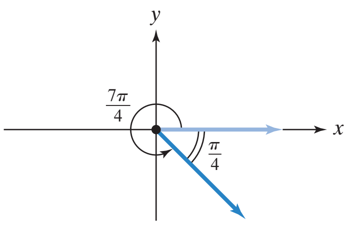


Figure 4.61 Reference angle for $\frac{7\pi}{4}$

The function value for the reference angle is $\sec\frac{\pi}{4} = \sqrt{2}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{7\pi}{4}$ lies in quadrant IV. Because the secant is positive in quadrant IV, we put a + sign before the function value of the reference angle. Thus,

$$\sec\left(-\frac{17\pi}{4}\right) = \sec\frac{7\pi}{4} = +\sec\frac{\pi}{4} = \sqrt{2}.$$

The secant is positive in quadrant IV.

The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$.

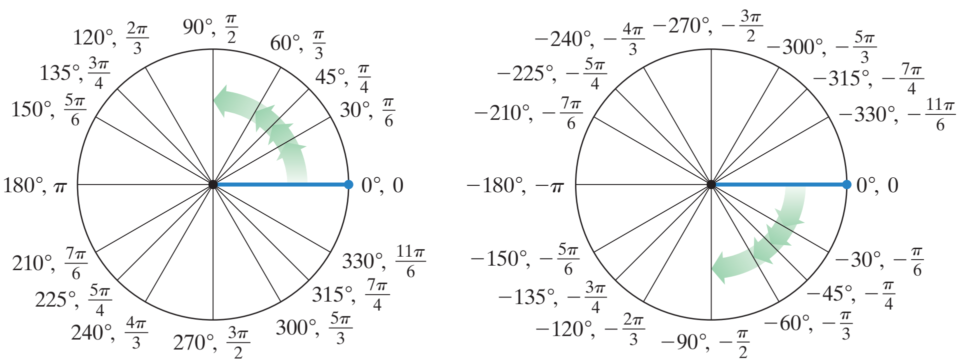
Check Point 8 Use reference angles to find the exact value of each of the following trigonometric functions:

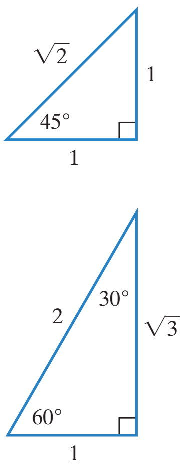
a. $\cos\frac{17\pi}{6}$ **b.** $\sin\left(-\frac{22\pi}{3}\right)$.

Study Tip

Evaluating trigonometric functions like those in Example 8 and Check Point 8 involves using a number of concepts, including finding coterminal angles and reference angles, locating special angles, determining the signs of trigonometric functions in specific quadrants, and finding the trigonometric functions of special angles $\left(30° = \frac{\pi}{6}, 45° = \frac{\pi}{4}, \text{ and } 60° = \frac{\pi}{3}\right)$. To be successful in trigonometry, it is often necessary to connect concepts. Here's an early reference sheet showing some of the concepts you should have at your fingertips (or memorized).

Degree and Radian Measures of Special and Quadrantal Angles

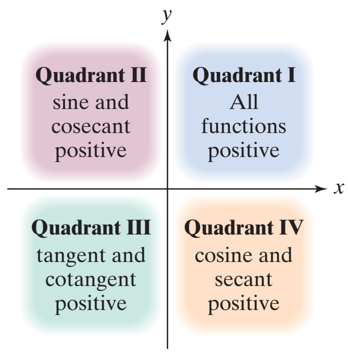




Special Right Triangles and Trigonometric Functions of Special Angles

θ	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Signs of the Trigonometric Functions



Trigonometric Functions of Quadrantal Angles

θ	$0° = 0$	$90° = \frac{\pi}{2}$	$180° = \pi$	$270° = \frac{3\pi}{2}$
$\sin\theta$	0	1	0	−1
$\cos\theta$	1	0	−1	0
$\tan\theta$	0	undefined	0	undefined

Using Reference Angles to Evaluate Trigonometric Functions

$\sin\theta = \square \sin\theta'$

$\cos\theta = \square \cos\theta'$

$\tan\theta = \square \tan\theta'$

+ or − in $\square$ determined by the quadrant in which θ lies and the sign of the function in that quadrant.

Exercise Set 4.4

Practice Exercises

In Exercises 1–8, a point on the terminal side of angle θ is given. Find the exact value of each of the six trigonometric functions of θ.

1. $(-4, 3)$ **2.** $(-12, 5)$ **3.** $(2, 3)$
4. $(3, 7)$ **5.** $(3, -3)$ **6.** $(5, -5)$
7. $(-2, -5)$ **8.** $(-1, -3)$

In Exercises 9–16, evaluate the trigonometric function at the quadrantal angle, or state that the expression is undefined.

9. $\cos \pi$ **10.** $\tan \pi$ **11.** $\sec \pi$
12. $\csc \pi$ **13.** $\tan \frac{3\pi}{2}$ **14.** $\cos \frac{3\pi}{2}$
15. $\cot \frac{\pi}{2}$ **16.** $\tan \frac{\pi}{2}$

In Exercises 17–22, let θ be an angle in standard position. Name the quadrant in which θ lies.

17. $\sin\theta > 0, \quad \cos\theta > 0$ **18.** $\sin\theta < 0, \quad \cos\theta > 0$
19. $\sin\theta < 0, \quad \cos\theta < 0$ **20.** $\tan\theta < 0, \quad \sin\theta < 0$
21. $\tan\theta < 0, \quad \cos\theta < 0$ **22.** $\cot\theta > 0, \quad \sec\theta < 0$

In Exercises 23–34, find the exact value of each of the remaining trigonometric functions of θ.

23. $\cos\theta = -\frac{3}{5}, \quad \theta$ in quadrant III
24. $\sin\theta = -\frac{12}{13}, \quad \theta$ in quadrant III
25. $\sin\theta = \frac{5}{13}, \quad \theta$ in quadrant II
26. $\cos\theta = \frac{4}{5}, \quad \theta$ in quadrant IV
27. $\cos\theta = \frac{8}{17}, \quad 270° < \theta < 360°$
28. $\cos\theta = \frac{1}{3}, \quad 270° < \theta < 360°$
29. $\tan\theta = -\frac{2}{3}, \quad \sin\theta > 0$ **30.** $\tan\theta = -\frac{1}{3}, \quad \sin\theta > 0$
31. $\tan\theta = \frac{4}{3}, \quad \cos\theta < 0$ **32.** $\tan\theta = \frac{5}{12}, \quad \cos\theta < 0$
33. $\sec\theta = -3, \quad \tan\theta > 0$ **34.** $\csc\theta = -4, \quad \tan\theta > 0$

In Exercises 35–60, find the reference angle for each angle.

35. $160°$ **36.** $170°$ **37.** $205°$
38. $210°$ **39.** $355°$ **40.** $351°$
41. $\frac{7\pi}{4}$ **42.** $\frac{5\pi}{4}$ **43.** $\frac{5\pi}{6}$
44. $\frac{5\pi}{7}$ **45.** $-150°$ **46.** $-250°$
47. $-335°$ **48.** $-359°$ **49.** 4.7
50. 5.5 **51.** $565°$ **52.** $553°$
53. $\frac{17\pi}{6}$ **54.** $\frac{11\pi}{4}$ **55.** $\frac{23\pi}{4}$
56. $\frac{17\pi}{3}$ **57.** $-\frac{11\pi}{4}$ **58.** $-\frac{17\pi}{6}$
59. $-\frac{25\pi}{6}$ **60.** $-\frac{13\pi}{3}$

In Exercises 61–86, use reference angles to find the exact value of each expression. Do not use a calculator.

61. $\cos 225°$ **62.** $\sin 300°$ **63.** $\tan 210°$
64. $\sec 240°$ **65.** $\tan 420°$ **66.** $\tan 405°$
67. $\sin \frac{2\pi}{3}$ **68.** $\cos \frac{3\pi}{4}$ **69.** $\csc \frac{7\pi}{6}$
70. $\cot \frac{7\pi}{4}$ **71.** $\tan \frac{9\pi}{4}$ **72.** $\tan \frac{9\pi}{2}$
73. $\sin(-240°)$ **74.** $\sin(-225°)$ **75.** $\tan\left(-\frac{\pi}{4}\right)$
76. $\tan\left(-\frac{\pi}{6}\right)$ **77.** $\sec 495°$ **78.** $\sec 510°$
79. $\cot \frac{19\pi}{6}$ **80.** $\cot \frac{13\pi}{3}$ **81.** $\cos \frac{23\pi}{4}$
82. $\cos \frac{35\pi}{6}$ **83.** $\tan\left(-\frac{17\pi}{6}\right)$ **84.** $\tan\left(-\frac{11\pi}{4}\right)$
85. $\sin\left(-\frac{17\pi}{3}\right)$ **86.** $\sin\left(-\frac{35\pi}{6}\right)$

Practice Plus

In Exercises 87–92, find the exact value of each expression. Write the answer as a single fraction. Do not use a calculator.

87. $\sin \frac{\pi}{3} \cos \pi - \cos \frac{\pi}{3} \sin \frac{3\pi}{2}$

88. $\sin \frac{\pi}{4} \cos 0 - \sin \frac{\pi}{6} \cos \pi$

89. $\sin \frac{11\pi}{4} \cos \frac{5\pi}{6} + \cos \frac{11\pi}{4} \sin \frac{5\pi}{6}$

90. $\sin \frac{17\pi}{3} \cos \frac{5\pi}{4} + \cos \frac{17\pi}{3} \sin \frac{5\pi}{4}$

91. $\sin \frac{3\pi}{2} \tan\left(-\frac{15\pi}{4}\right) - \cos\left(-\frac{5\pi}{3}\right)$

92. $\sin \frac{3\pi}{2} \tan\left(-\frac{8\pi}{3}\right) + \cos\left(-\frac{5\pi}{6}\right)$

In Exercises 93–98, let

$$f(x) = \sin x, \; g(x) = \cos x, \text{ and } h(x) = 2x.$$

Find the exact value of each expression. Do not use a calculator.

93. $f\left(\frac{4\pi}{3} + \frac{\pi}{6}\right) + f\left(\frac{4\pi}{3}\right) + f\left(\frac{\pi}{6}\right)$

94. $g\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) + g\left(\frac{5\pi}{6}\right) + g\left(\frac{\pi}{6}\right)$

95. $(h \circ g)\left(\frac{17\pi}{3}\right)$ **96.** $(h \circ f)\left(\frac{11\pi}{4}\right)$

97. the average rate of change of f from $x_1 = \frac{5\pi}{4}$ to $x_2 = \frac{3\pi}{2}$

98. the average rate of change of g from $x_1 = \frac{3\pi}{4}$ to $x_2 = \pi$

In Exercises 99–104, find two values of θ, $0 \le \theta < 2\pi$, that satisfy each equation.

99. $\sin\theta = \frac{\sqrt{2}}{2}$ **100.** $\cos\theta = \frac{1}{2}$

101. $\sin\theta = -\frac{\sqrt{2}}{2}$ **102.** $\cos\theta = -\frac{1}{2}$

103. $\tan\theta = -\sqrt{3}$ **104.** $\tan\theta = -\frac{\sqrt{3}}{3}$

Writing in Mathematics

105. If you are given a point on the terminal side of angle θ, explain how to find $\sin\theta$.

106. Explain why $\tan 90°$ is undefined.

107. If $\cos\theta > 0$ and $\tan\theta < 0$, explain how to find the quadrant in which θ lies.

108. What is a reference angle? Give an example with your description.

109. Explain how reference angles are used to evaluate trigonometric functions. Give an example with your description.

Critical Thinking Exercises

Make Sense? *In Exercises 110–113, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

110. I'm working with a quadrantal angle θ for which $\sin\theta$ is undefined.

111. This angle θ is in a quadrant in which $\sin\theta < 0$ and $\csc\theta > 0$.

112. I am given that $\tan\theta = \frac{3}{5}$, so I can conclude that $y = 3$ and $x = 5$.

113. When I found the exact value of $\cos\frac{14\pi}{3}$, I used a number of concepts, including coterminal angles, reference angles, finding the cosine of a special angle, and knowing the cosine's sign in various quadrants.

Preview Exercises

Exercises 114–116 will help you prepare for the material covered in the next section. In each exercise, complete the table of coordinates. Do not use a calculator.

114. $y = \frac{1}{2}\cos(4x + \pi)$

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
y					

115. $y = 4\sin\left(2x - \frac{2\pi}{3}\right)$

x	$\frac{\pi}{3}$	$\frac{7\pi}{12}$	$\frac{5\pi}{6}$	$\frac{13\pi}{12}$	$\frac{4\pi}{3}$
y					

116. $y = 3\sin\frac{\pi}{2}x$

x	0	$\frac{1}{3}$	1	$\frac{5}{3}$	2	$\frac{7}{3}$	3	$\frac{11}{3}$	4
y									

After completing this table of coordinates, plot the nine ordered pairs as points in a rectangular coordinate system. Then connect the points with a smooth curve.

Chapter 4 Mid-Chapter Check Point

What You Know: We learned to use radians to measure angles: One radian (approximately 57°) is the measure of the central angle that intercepts an arc equal in length to the radius of the circle. Using $180° = \pi$ radians, we converted degrees to radians $\left(\text{multiply by } \frac{\pi}{180°}\right)$ and radians to degrees $\left(\text{multiply by } \frac{180°}{\pi}\right)$. We defined the six trigonometric functions using coordinates of points along the unit circle, right triangles, and angles in standard position. Evaluating trigonometric functions using reference angles involved connecting a number of concepts, including finding coterminal and reference angles, locating special angles, determining the signs of the trigonometric functions in specific quadrants, and finding the function values at special angles. Use the important Study Tip on page 512 as a reference sheet to help connect these concepts.

In Exercises 1–2, convert each angle in degrees to radians. Express your answer as a multiple of π.

1. 10°   **2.** $-105°$

In Exercises 3–4, convert each angle in radians to degrees.

3. $\frac{5\pi}{12}$ **4.** $-\frac{13\pi}{20}$

In Exercises 5–7,

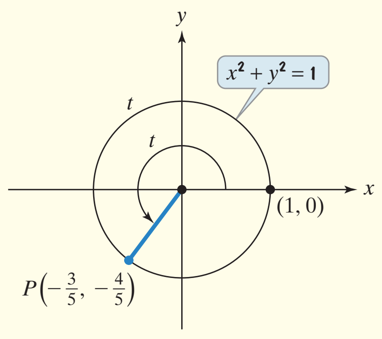
a. *Find a positive angle less than 360° or 2π that is coterminal with the given angle.*

b. *Draw the given angle in standard position.*

c. *Find the reference angle for the given angle.*

5. $\frac{11\pi}{3}$ **6.** $-\frac{19\pi}{4}$ **7.** 510°

8. Use the point shown on the unit circle to find each of the six trigonometric functions at t.



9. Use the triangle to find each of the six trigonometric functions of θ.