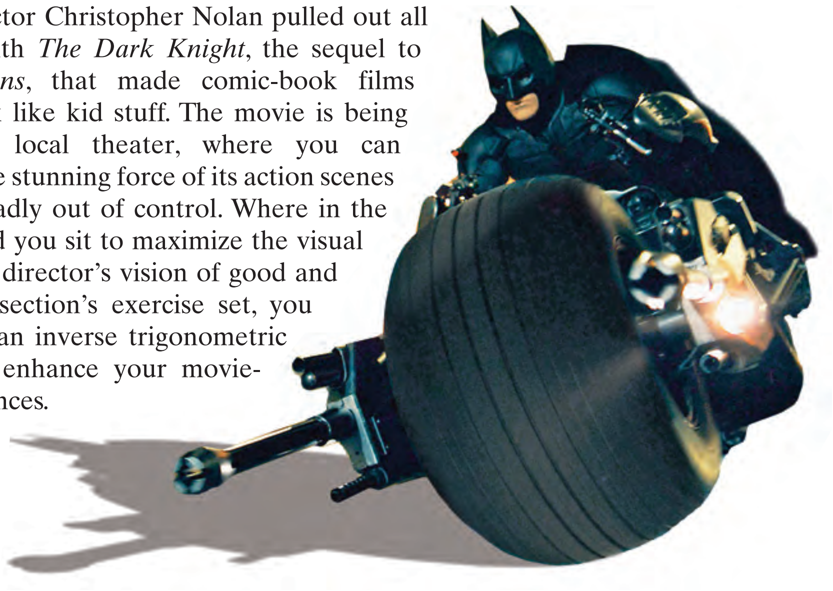
Section 4.7

Inverse Trigonometric Functions

Objectives

1. Understand and use the inverse sine function.
2. Understand and use the inverse cosine function.
3. Understand and use the inverse tangent function.
4. Use a calculator to evaluate inverse trigonometric functions.
5. Find exact values of composite functions with inverse trigonometric functions.

In 2008, director Christopher Nolan pulled out all the stops with *The Dark Knight*, the sequel to *Batman Begins*, that made comic-book films before it look like kid stuff. The movie is being shown at a local theater, where you can experience the stunning force of its action scenes that teeter madly out of control. Where in the theater should you sit to maximize the visual impact of the director's vision of good and evil? In this section's exercise set, you will see how an inverse trigonometric function can enhance your movie-going experiences.



Study Tip

Here are some helpful things to remember from our earlier discussion of inverse functions.

- If no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function.
- If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of the inverse function, denoted f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

1. Understand and use the inverse sine function.

The Inverse Sine Function

Figure 4.86 shows the graph of $y = \sin x$. Can you see that every horizontal line that can be drawn between -1 and 1 intersects the graph infinitely many times? Thus, the sine function is not one-to-one and has no inverse function.

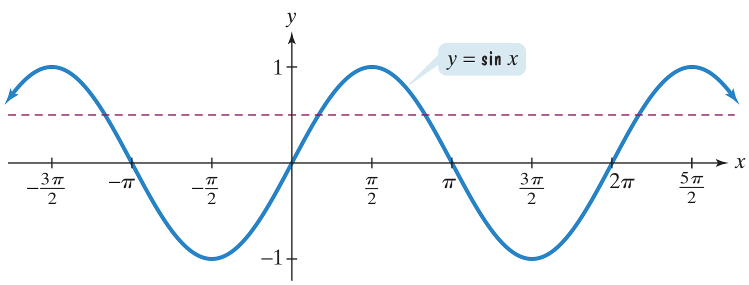


Figure 4.86 The horizontal line test shows that the sine function is not one-to-one and has no inverse function.

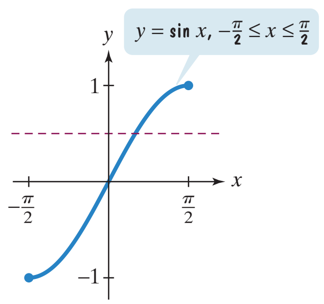


Figure 4.87 The restricted sine function passes the horizontal line test. It is one-to-one and has an inverse function.

In **Figure 4.87**, we have taken a portion of the sine curve, restricting the domain of the sine function to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. With this restricted domain, every horizontal line that can be drawn between -1 and 1 intersects the graph exactly once. Thus, the restricted function passes the horizontal line test and is one-to-one.

On the restricted domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y = \sin x$ has an inverse function. The inverse of the restricted sine function is called the **inverse sine function**. Two notations are commonly used to denote the inverse sine function:

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x.$$

In this book, we will use $y = \sin^{-1} x$. This notation has the same symbol as the inverse function notation $f^{-1}(x)$.

The Inverse Sine Function

The **inverse sine function**, denoted by $\sin^{-1}$, is the inverse of the restricted sine function $y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Thus,

$$y = \sin^{-1} x \quad \text{means} \quad \sin y = x,$$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $-1 \le x \le 1$. We read $y = \sin^{-1} x$ as "y equals the inverse sine at x."

Study Tip

The notation $y = \sin^{-1} x$ does not mean $y = \frac{1}{\sin x}$. The notation $y = \frac{1}{\sin x}$, or the reciprocal of the sine function, is written $y = (\sin x)^{-1}$ and means $y = \csc x$.

Inverse sine function: $y = \sin^{-1} x$

Reciprocal of sine function: $y = (\sin x)^{-1} = \frac{1}{\sin x} = \csc x$

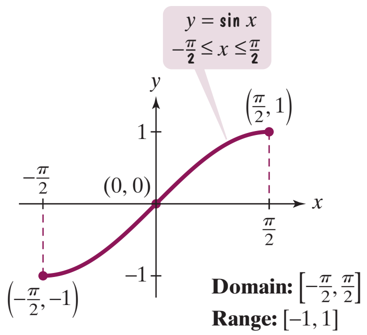


Figure 4.88 The restricted sine function

One way to graph $y = \sin^{-1} x$ is to take points on the graph of the restricted sine function and reverse the order of the coordinates. For example, **Figure 4.88** shows that $\left(-\frac{\pi}{2}, -1\right)$, $(0, 0)$, and $\left(\frac{\pi}{2}, 1\right)$ are on the graph of the restricted sine function. Reversing the order of the coordinates gives $\left(-1, -\frac{\pi}{2}\right)$, $(0, 0)$, and $\left(1, \frac{\pi}{2}\right)$. We now use these three points to sketch the inverse sine function. The graph of $y = \sin^{-1} x$ is shown in Figure **4.89**.

Another way to obtain the graph of $y = \sin^{-1} x$ is to reflect the graph of the restricted sine function about the line $y = x$, shown in Figure **4.90**. The red graph is the restricted sine function and the blue graph is the graph of $y = \sin^{-1} x$.

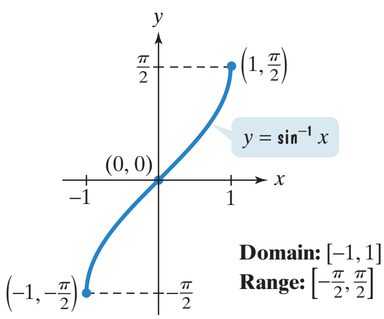


Figure 4.89 The graph of the inverse sine function

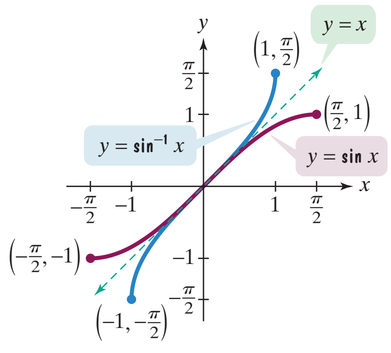


Figure 4.90 Using a reflection to obtain the graph of the inverse sine function

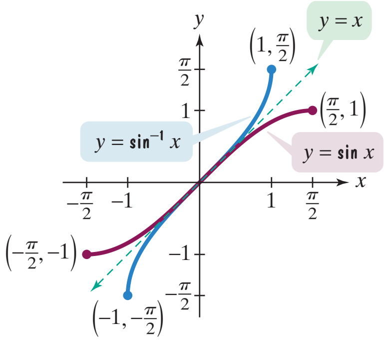


Figure 4.90 (repeated)

Exact values of $\sin^{-1} x$ can be found by thinking of **$\sin^{-1} x$ as the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.** For example, we can use the two points on the blue graph of the inverse sine function in **Figure 4.90** to write

$$\sin^{-1}(-1) = -\frac{\pi}{2} \quad \text{and} \quad \sin^{-1} 1 = \frac{\pi}{2}.$$

The angle whose sine is −1 is $-\frac{\pi}{2}$.

The angle whose sine is 1 is $\frac{\pi}{2}$.

Because we are thinking of $\sin^{-1} x$ in terms of an angle, we will represent such an angle by θ.

Finding Exact Values of $\sin^{-1} x$

1. Let $\theta = \sin^{-1} x$.

2. Rewrite $\theta = \sin^{-1} x$ as $\sin \theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

3. Use the exact values in **Table 4.7** to find the value of θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin \theta = x$.

Table 4.7 **Exact Values for $\sin \theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$**

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

EXAMPLE 1 Finding the Exact Value of an Inverse Sine Function

Find the exact value of $\sin^{-1} \frac{\sqrt{2}}{2}$.

Solution

Step 1 Let $\theta = \sin^{-1} x$. Thus,

$$\theta = \sin^{-1} \frac{\sqrt{2}}{2}.$$

We must find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.

Step 2 Rewrite $\theta = \sin^{-1} x$ as $\sin \theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Using the definition of the inverse sine function, we rewrite $\theta = \sin^{-1} \frac{\sqrt{2}}{2}$ as

$$\sin \theta = \frac{\sqrt{2}}{2}, \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Step 3 Use the exact values in Table 4.7 to find the value of θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin \theta = x$. **Table 4.7** shows that the only angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin \theta = \frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Thus, $\theta = \frac{\pi}{4}$. Because θ, in step 1, represents $\sin^{-1} \frac{\sqrt{2}}{2}$, we conclude that

$$\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}.$$

The angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$.

Check Point 1 Find the exact value of $\sin^{-1}\frac{\sqrt{3}}{2}$.

EXAMPLE 2 Finding the Exact Value of an Inverse Sine Function

Find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution

Step 1 Let $\theta = \sin^{-1} x$. Thus,

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right).$$

We must find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

Step 2 Rewrite $\theta = \sin^{-1} x$ as $\sin\theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We rewrite $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$ and obtain

$$\sin\theta = -\frac{1}{2}, \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Step 3 Use the exact values in Table 4.7 to find the value of θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin\theta = x$. Table 4.7 shows that the only angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin\theta = -\frac{1}{2}$ is $-\frac{\pi}{6}$. Thus,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Check Point 2 Find the exact value of $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Some inverse sine expressions cannot be evaluated. Because the domain of the inverse sine function is $[-1, 1]$, it is only possible to evaluate $\sin^{-1}x$ for values of x in this domain. Thus, $\sin^{-1}3$ cannot be evaluated. There is no angle whose sine is 3.

2 Understand and use the inverse cosine function.

The Inverse Cosine Function

Figure 4.91 shows how we restrict the domain of the cosine function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $[0, \pi]$, shown by the dark blue graph. Over this interval, the restricted cosine function passes the horizontal line test and has an inverse function.

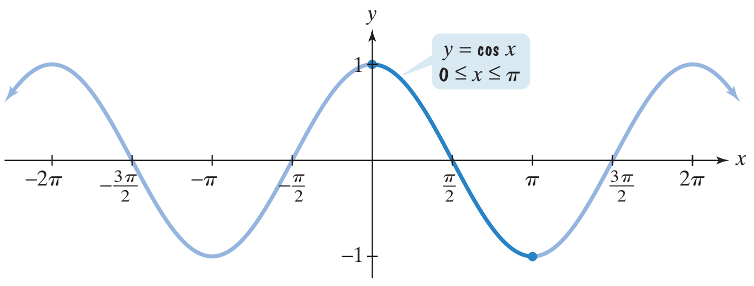


Figure 4.91 $y = \cos x$ is one-to-one on the interval $[0, \pi]$.

The Inverse Cosine Function

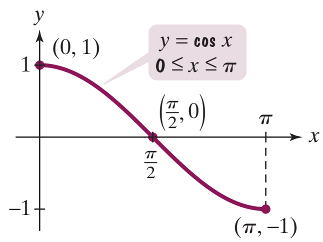
The **inverse cosine function**, denoted by $\cos^{-1}$, is the inverse of the restricted cosine function $y = \cos x$, $0 \le x \le \pi$. Thus,

$$y = \cos^{-1} x \quad \text{means} \quad \cos y = x,$$

where $0 \le y \le \pi$ and $-1 \le x \le 1$.

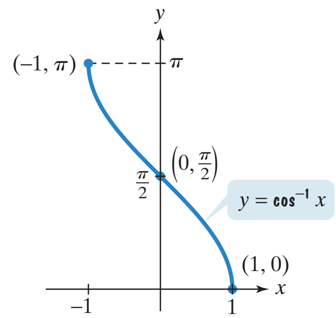
One way to graph $y = \cos^{-1} x$ is to take points on the graph of the restricted cosine function and reverse the order of the coordinates. For example, **Figure 4.92** shows that $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, and $(\pi, -1)$ are on the graph of the restricted cosine function. Reversing the order of the coordinates gives $(1, 0)$, $\left(0, \frac{\pi}{2}\right)$, and $(-1, \pi)$.

We now use these three points to sketch the inverse cosine function. The graph of $y = \cos^{-1} x$ is shown in **Figure 4.93**. You can also obtain this graph by reflecting the graph of the restricted cosine function about the line $y = x$.



Domain: $[0, \pi]$
Range: $[-1, 1]$

Figure 4.92 The restricted cosine function



Domain: $[-1, 1]$
Range: $[0, \pi]$

Figure 4.93 The graph of the inverse cosine function

Exact values of $\cos^{-1} x$ can be found by thinking of **$\cos^{-1} x$ as the angle in the interval $[0, \pi]$ whose cosine is x.**

Finding Exact Values of $\cos^{-1} x$

1. Let $\theta = \cos^{-1} x$.
2. Rewrite $\theta = \cos^{-1} x$ as $\cos \theta = x$, where $0 \le \theta \le \pi$.
3. Use the exact values in **Table 4.8** to find the value of θ in $[0, \pi]$ that satisfies $\cos \theta = x$.

Table 4.8 Exact Values for $\cos \theta$, $0 \le \theta \le \pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

EXAMPLE 3 Finding the Exact Value of an Inverse Cosine Function

Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Solution

Step 1 Let $\theta = \cos^{-1} x$. Thus,

$$\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right).$$

We must find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

Step 2 Rewrite $\theta = \cos^{-1} x$ as $\cos\theta = x$, where $0 \le \theta \le \pi$. We obtain

$$\cos\theta = -\frac{\sqrt{3}}{2}, \text{ where } 0 \le \theta \le \pi.$$

Step 3 Use the exact values in Table 4.8 to find the value of θ in $[0, \pi]$ that satisfies $\cos\theta = x$. The table on the previous page shows that the only angle in the interval $[0, \pi]$ that satisfies $\cos\theta = -\frac{\sqrt{3}}{2}$ is $\frac{5\pi}{6}$. Thus, $\theta = \frac{5\pi}{6}$ and

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

The angle in $[0, \pi]$ whose cosine is $-\frac{\sqrt{3}}{2}$ is $\frac{5\pi}{6}$.

Check Point 3 Find the exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

3 Understand and use the inverse tangent function.

The Inverse Tangent Function

Figure 4.94 shows how we restrict the domain of the tangent function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, shown by the solid blue graph. Over this interval, the restricted tangent function passes the horizontal line test and has an inverse function.

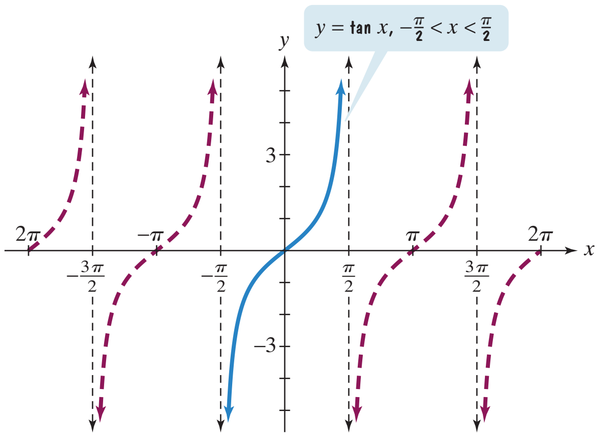


Figure 4.94 $y = \tan x$ is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The Inverse Tangent Function

The **inverse tangent function**, denoted by $\tan^{-1}$, is the inverse of the restricted tangent function $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Thus,

$$y = \tan^{-1} x \quad \text{means} \quad \tan y = x,$$

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < \infty$.

We graph $y = \tan^{-1} x$ by taking points on the graph of the restricted function and reversing the order of the coordinates. **Figure 4.95** shows that $\left(-\frac{\pi}{4}, -1\right)$, (0, 0), and $\left(\frac{\pi}{4}, 1\right)$ are on the graph of the restricted tangent function. Reversing the order gives $\left(-1, -\frac{\pi}{4}\right)$, (0, 0), and $\left(1, \frac{\pi}{4}\right)$. We now use these three points to graph the inverse tangent function. The graph of $y = \tan^{-1} x$ is shown in **Figure 4.96**. Notice that the vertical asymptotes become horizontal asymptotes for the graph of the inverse function.

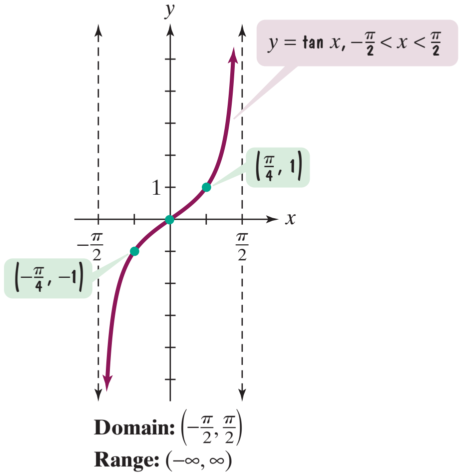


Figure 4.95 The restricted tangent function

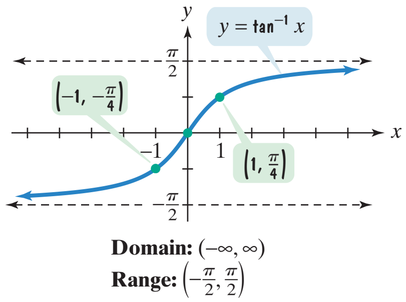


Figure 4.96 The graph of the inverse tangent function

Exact values of $\tan^{-1} x$ can be found by thinking of **$\tan^{-1} x$ as the angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.**

Finding Exact Values of $\tan^{-1} x$

1. Let $\theta = \tan^{-1} x$.
2. Rewrite $\theta = \tan^{-1} x$ as $\tan \theta = x$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
3. Use the exact values in **Table 4.9** to find the value of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfies $\tan \theta = x$.

Table 4.9 Exact Values for $\tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan \theta$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

EXAMPLE 4 Finding the Exact Value of an Inverse Tangent Function

Find the exact value of $\tan^{-1}\sqrt{3}$.

Solution

Step 1 Let $\theta = \tan^{-1} x$. Thus,

$$\theta = \tan^{-1}\sqrt{3}.$$

We must find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

Step 2 Rewrite $\theta = \tan^{-1} x$ as $\tan\theta = x$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We obtain

$$\tan\theta = \sqrt{3}, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Step 3 Use the exact values in Table 4.9 to find the value of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfies $\tan\theta = x$. The table on the previous page shows that the only angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfies $\tan\theta = \sqrt{3}$ is $\frac{\pi}{3}$. Thus, $\theta = \frac{\pi}{3}$ and

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}.$$

The angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $\sqrt{3}$ is $\frac{\pi}{3}$.

Study Tip

Do not confuse the domains of the restricted trigonometric functions with the intervals on which the nonrestricted functions complete one cycle.

Trigonometric Function	Domain of Restricted Function	Interval on Which Nonrestricted Function's Graph Completes One Period
$y = \sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, 2\pi]$
$y = \cos x$	$[0, \pi]$	$[0, 2\pi]$
$y = \tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Period: 2π

Period: 2π

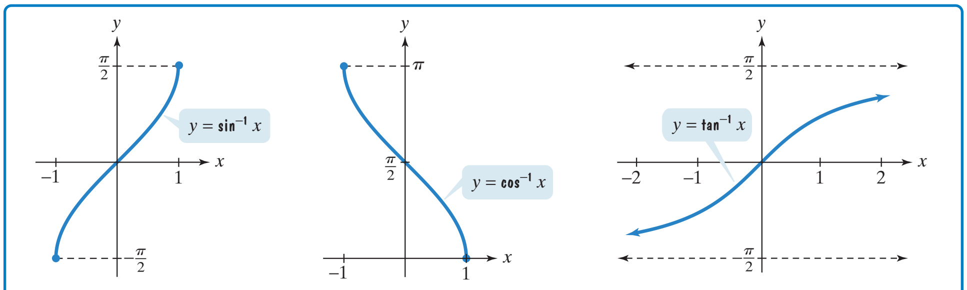
Period: π

These domain restrictions are the range for $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$, respectively.

Check Point 4 Find the exact value of $\tan^{-1}(-1)$.

Table 4.10 summarizes the graphs of the three basic inverse trigonometric functions. Below each of the graphs is a description of the function's domain and range.

Table 4.10 Graphs of the Three Basic Inverse Trigonometric Functions



Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain: $[-1, 1]$
Range: $[0, \pi]$

Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4 Use a calculator to evaluate inverse trigonometric functions.

Using a Calculator to Evaluate Inverse Trigonometric Functions

Calculators give approximate values of inverse trigonometric functions. Use the secondary keys marked $\boxed{\text{SIN}^{-1}}$, $\boxed{\text{COS}^{-1}}$, and $\boxed{\text{TAN}^{-1}}$. These keys are not buttons that you actually press. They are the secondary functions for the buttons labeled $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, and $\boxed{\text{TAN}}$, respectively. Consult your manual for the location of this feature.

EXAMPLE 5 Calculators and Inverse Trigonometric Functions

Use a calculator to find the value to four decimal places of each function:

a. $\sin^{-1}\frac{1}{4}$ **b.** $\tan^{-1}(-9.65)$.

Solution

Scientific Calculator Solution

Function	Mode	Keystrokes	Display, Rounded to Four Places
a. $\sin^{-1}\frac{1}{4}$	Radian	1 $\boxed{\div}$ 4 $\boxed{=}$ $\boxed{\text{2nd}}$ $\boxed{\text{SIN}}$	0.2527
b. $\tan^{-1}(-9.65)$	Radian	9.65 $\boxed{+/-}$ $\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$	-1.4675

Graphing Calculator Solution

Function	Mode	Keystrokes	Display, Rounded to Four Places
a. $\sin^{-1}\frac{1}{4}$	Radian	$\boxed{\text{2nd}}$ $\boxed{\text{SIN}}$ $\boxed{(}$ 1 $\boxed{\div}$ $\boxed{4}$ $\boxed{)}$ $\boxed{\text{ENTER}}$	0.2527
b. $\tan^{-1}(-9.65)$	Radian	$\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$ $\boxed{(-)}$ 9.65 $\boxed{\text{ENTER}}$	-1.4675

Check Point 5 Use a calculator to find the value to four decimal places of each function:

a. $\cos^{-1}\frac{1}{3}$ **b.** $\tan^{-1}(-35.85)$.

What happens if you attempt to evaluate an inverse trigomometric function at a value that is not in its domain? In real number mode, most calculators will display an error message. For example, an error message can result if you attempt to approximate $\cos^{-1} 3$. There is no angle whose cosine is 3. The domain of the inverse cosine function is $[-1, 1]$ and 3 does not belong to this domain.

5 Find exact values of composite functions with inverse trigonometric functions.

Composition of Functions Involving Inverse Trigonometric Functions

In our earlier discussion of functions and their inverses, we saw that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

x must be in the domain of f^{-1}. x must be in the domain of f.

We apply these properties to the sine, cosine, tangent, and their inverse functions to obtain the following properties:

Inverse Properties

The Sine Function and Its Inverse

$\sin(\sin^{-1} x) = x$ for every x in the interval $[-1, 1]$

$\sin^{-1}(\sin x) = x$ for every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Cosine Function and Its Inverse

$\cos(\cos^{-1} x) = x$ for every x in the interval $[-1, 1]$

$\cos^{-1}(\cos x) = x$ for every x in the interval $[0, \pi]$

The Tangent Function and Its Inverse

$\tan(\tan^{-1} x) = x$ for every real number x

$\tan^{-1}(\tan x) = x$ for every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The restrictions on x in the inverse properties are a bit tricky. For example,

$$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

$\sin^{-1}(\sin x) = x$ for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Observe that $\frac{\pi}{4}$ is in this interval.

Can we use $\sin^{-1}(\sin x) = x$ to find the exact value of $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$? Is $\frac{5\pi}{4}$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$? No. Thus, to evaluate $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$, we must first find $\sin\frac{5\pi}{4}$.

$\frac{5\pi}{4}$ is in quadrant III, where the sine is negative.

$$\sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$.

We evaluate $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$ as follows:

$$\sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

If necessary, see Table 4.7 on page 552.

To determine how to evaluate the composition of functions involving inverse trigonometric functions, first examine the value of x. You can use the inverse properties in the box only if x is in the specified interval.

EXAMPLE 6 Evaluating Compositions of Functions and Their Inverses

Find the exact value, if possible:

a. $\cos(\cos^{-1} 0.6)$ **b.** $\sin^{-1}\left(\sin \frac{3\pi}{2}\right)$ **c.** $\cos(\cos^{-1} 1.5)$.

Solution

a. The inverse property $\cos(\cos^{-1}x) = x$ applies for every x in $[-1, 1]$. To evaluate $\cos(\cos^{-1} 0.6)$, observe that $x = 0.6$. This value of x lies in $[-1, 1]$, which is the domain of the inverse cosine function. This means that we can use the inverse property $\cos(\cos^{-1}x) = x$. Thus,

$$\cos(\cos^{-1} 0.6) = 0.6.$$

b. The inverse property $\sin^{-1}(\sin x) = x$ applies for every x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. To evaluate $\sin^{-1}\left(\sin \frac{3\pi}{2}\right)$, observe that $x = \frac{3\pi}{2}$. This value of x does not lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. To evaluate this expression, we first find $\sin \frac{3\pi}{2}$.

$$\sin^{-1}\left(\sin \frac{3\pi}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$$

The angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is -1 is $-\frac{\pi}{2}$.

c. The inverse property $\cos(\cos^{-1} x) = x$ applies for every x in $[-1, 1]$. To attempt to evaluate $\cos(\cos^{-1} 1.5)$, observe that $x = 1.5$. This value of x does not lie in $[-1, 1]$, which is the domain of the inverse cosine function. Thus, the expression $\cos(\cos^{-1} 1.5)$ is not defined because $\cos^{-1} 1.5$ is not defined.

Check Point 6 Find the exact value, if possible:

a. $\cos(\cos^{-1} 0.7)$ **b.** $\sin^{-1}(\sin \pi)$ **c.** $\cos[\cos^{-1}(-1.2)]$.

We can use points on terminal sides of angles in standard position to find exact values of expressions involving the composition of a function and a different inverse function. Here are two examples:

$$\cos\left(\tan^{-1} \frac{5}{12}\right) \qquad \cot\left[\sin^{-1}\left(-\frac{1}{3}\right)\right].$$

Inner part involves the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $\frac{5}{12}$.

Inner part involves the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{3}$.

The inner part of each expression involves an angle. To evaluate such expressions, we represent such angles by θ. Then we use a sketch that illustrates our representation. Examples 7 and 8 show how to carry out such evaluations.

EXAMPLE 7 Evaluating a Composite Trigonometric Expression

Find the exact value of $\cos\left(\tan^{-1} \frac{5}{12}\right)$.

Solution

We let θ represent the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $\frac{5}{12}$. Thus,

$$\theta = \tan^{-1} \frac{5}{12}.$$

We are looking for the exact value of $\cos\left(\tan^{-1}\frac{5}{12}\right)$, with $\theta = \tan^{-1}\frac{5}{12}$. Using the definition of the inverse tangent function, we can rewrite $\theta = \tan^{-1}\frac{5}{12}$ as

$$\tan\theta = \frac{5}{12}, \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

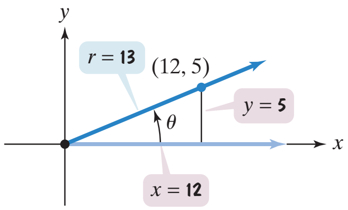


Figure 4.97 Representing $\tan\theta = \frac{5}{12}$

Because tan θ is positive, θ must be an angle in $\left(0, \frac{\pi}{2}\right)$. Thus, θ is a first-quadrant angle. **Figure 4.97** shows a right triangle in quadrant I with

$$\tan\theta = \frac{5}{12}.$$

Side opposite θ, or y

Side adjacent to θ, or x

The hypotenuse of the triangle, r, or the distance from the origin to (12, 5), is found using $r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{x^2 + y^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$$

We use the values for x and r to find the exact value of $\cos\left(\tan^{-1}\frac{5}{12}\right)$.

$$\cos\left(\tan^{-1}\frac{5}{12}\right) = \cos\theta = \frac{\text{side adjacent to } \theta, \text{ or } x}{\text{hypotenuse, or } r} = \frac{12}{13}$$

Check Point 7 Find the exact value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

EXAMPLE 8 Evaluating a Composite Trigonometric Expression

Find the exact value of $\cot\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$.

Solution We let θ represent the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{3}$. Thus,

$$\theta = \sin^{-1}\left(-\frac{1}{3}\right) \quad \text{and} \quad \sin\theta = -\frac{1}{3}, \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

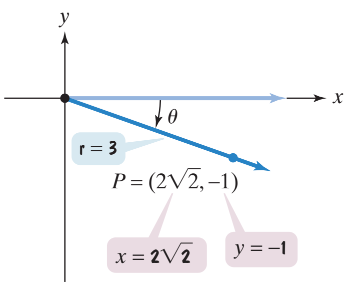


Figure 4.98 Representing $\sin\theta = -\frac{1}{3}$

Because $\sin\theta$ is negative in $\sin\theta = -\frac{1}{3}$, θ must be an angle in $\left[-\frac{\pi}{2}, 0\right)$. Thus, θ is a negative angle that lies in quadrant IV. **Figure 4.98** shows angle θ in quadrant IV with

In quadrant IV, y is negative.

$$\sin\theta = -\frac{1}{3} = \frac{y}{r} = \frac{-1}{3}.$$

Thus, $y = -1$ and $r = 3$. The value of x can be found using $r = \sqrt{x^2 + y^2}$ or $x^2 + y^2 = r^2$.

$x^2 + (-1)^2 = 3^2$ — Use $x^2 + y^2 = r^2$ with $y = -1$ and $r = 3$.

$x^2 + 1 = 9$ — Square −1 and square 3.

$x^2 = 8$ — Subtract 1 from both sides.

$x = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$ — Use the square root property. Remember that x is positive in quadrant IV.

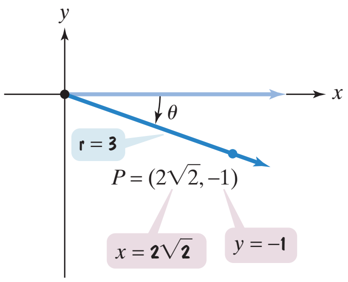


Figure 4.98 (repeated)
Representing $\sin\theta = -\frac{1}{3}$

We use $x = 2\sqrt{2}$ and $y = -1$ to find the exact value of $\cot\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$.

$$\cot\left[\sin^{-1}\left(-\frac{1}{3}\right)\right] = \cot\theta = \frac{x}{y} = \frac{2\sqrt{2}}{-1} = -2\sqrt{2}$$

Check Point 8 Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Some composite functions with inverse trigonometric functions can be simplified to algebraic expressions. To simplify such an expression, we represent the inverse trigonometric function in the expression by θ. Then we use a right triangle.

EXAMPLE 9 Simplifying an Expression Involving $\sin^{-1} x$

If $0 < x \le 1$, write $\cos(\sin^{-1} x)$ as an algebraic expression in x.

Solution We let θ represent the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. Thus,

$$\theta = \sin^{-1} x \quad \text{and} \quad \sin\theta = x, \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Because $0 < x \le 1$, $\sin\theta$ is positive. Thus, θ is a first-quadrant angle and can be represented as an acute angle of a right triangle. **Figure 4.99** shows a right triangle with

$$\sin\theta = x = \frac{x}{1}.$$

Side opposite θ

Hypotenuse

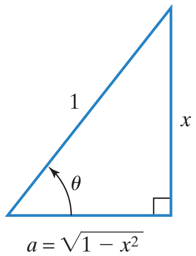


Figure 4.99 Representing $\sin\theta = x$

The third side, a in **Figure 4.99**, can be found using the Pythagorean Theorem.

$$a^2 + x^2 = 1^2$$ Apply the Pythagorean Theorem to the right triangle in Figure 4.99.

$$a^2 = 1 - x^2$$ Subtract x^2 from both sides.

$$a = \sqrt{1 - x^2}$$ Use the square root property and solve for a. Remember that side a is positive.

We use the right triangle in **Figure 4.99** to write $\cos(\sin^{-1} x)$ as an algebraic expression.

$$\cos(\sin^{-1} x) = \cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}$$

Check Point 9 If $x > 0$, write $\sec(\tan^{-1} x)$ as an algebraic expression in x.

The inverse secant function, $y = \sec^{-1} x$, is used in calculus. However, inverse cotangent and inverse cosecant functions are rarely used. Two of these remaining inverse trigonometric functions are briefly developed in the exercise set that follows.

Exercise Set 4.7

Practice Exercises

In Exercises 1–18, find the exact value of each expression.

1. $\sin^{-1}\frac{1}{2}$

2. $\sin^{-1} 0$

3. $\sin^{-1}\frac{\sqrt{2}}{2}$

4. $\sin^{-1}\frac{\sqrt{3}}{2}$

5. $\sin^{-1}\left(-\frac{1}{2}\right)$

6. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

7. $\cos^{-1}\frac{\sqrt{3}}{2}$

8. $\cos^{-1}\frac{\sqrt{2}}{2}$

9. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

10. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

11. $\cos^{-1} 0$

12. $\cos^{-1} 1$

13. $\tan^{-1}\frac{\sqrt{3}}{3}$

14. $\tan^{-1} 1$

15. $\tan^{-1} 0$

16. $\tan^{-1}(-1)$

17. $\tan^{-1}(-\sqrt{3})$

18. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

In Exercises 19–30, use a calculator to find the value of each expression rounded to two decimal places.

19. $\sin^{-1} 0.3$

20. $\sin^{-1} 0.47$

21. $\sin^{-1}(-0.32)$

22. $\sin^{-1}(-0.625)$

23. $\cos^{-1}\frac{3}{8}$

24. $\cos^{-1}\frac{4}{9}$

25. $\cos^{-1}\frac{\sqrt{5}}{7}$

26. $\cos^{-1}\frac{\sqrt{7}}{10}$

27. $\tan^{-1}(-20)$

28. $\tan^{-1}(-30)$

29. $\tan^{-1}(-\sqrt{473})$

30. $\tan^{-1}(-\sqrt{5061})$

In Exercises 31–46, find the exact value of each expression, if possible. Do not use a calculator.

31. $\sin(\sin^{-1} 0.9)$

32. $\cos(\cos^{-1} 0.57)$

33. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

34. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

35. $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

36. $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

37. $\tan(\tan^{-1} 125)$

38. $\tan(\tan^{-1} 380)$

39. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

40. $\tan^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right]$

41. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

42. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

43. $\sin^{-1}(\sin\pi)$

44. $\cos^{-1}(\cos 2\pi)$

45. $\sin(\sin^{-1}\pi)$

46. $\cos^{-1}(\cos^{-1} 3\pi)$

In Exercises 47–62, use a sketch to find the exact value of each expression.

47. $\cos\left(\sin^{-1}\frac{4}{5}\right)$

48. $\sin\left(\tan^{-1}\frac{7}{24}\right)$

49. $\tan\left(\cos^{-1}\frac{5}{13}\right)$

50. $\cot\left(\sin^{-1}\frac{5}{13}\right)$

51. $\tan\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$

52. $\cos\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]$

53. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

54. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

55. $\sec\left[\sin^{-1}\left(-\frac{1}{4}\right)\right]$

56. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

57. $\tan\left[\cos^{-1}\left(-\frac{1}{3}\right)\right]$

58. $\tan\left[\cos^{-1}\left(-\frac{1}{4}\right)\right]$

59. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

60. $\sec\left[\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right]$

61. $\cos\left[\tan^{-1}\left(-\frac{2}{3}\right)\right]$

62. $\sin\left[\tan^{-1}\left(-\frac{3}{4}\right)\right]$

In Exercises 63–72, use a right triangle to write each expression as an algebraic expression. Assume that x is positive and that the given inverse trigonometric function is defined for the expression in x.

63. $\tan(\cos^{-1} x)$

64. $\sin(\tan^{-1} x)$

65. $\cos(\sin^{-1} 2x)$

66. $\sin(\cos^{-1} 2x)$

67. $\cos\left(\sin^{-1}\frac{1}{x}\right)$

68. $\sec\left(\cos^{-1}\frac{1}{x}\right)$

69. $\cot\left(\tan^{-1}\frac{x}{\sqrt{3}}\right)$

70. $\cot\left(\tan^{-1}\frac{x}{\sqrt{2}}\right)$

71. $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)$

72. $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)$

73. **a.** Graph the restricted secant function, $y = \sec x$, by restricting x to the intervals $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right]$.

b. Use the horizontal line test to explain why the restricted secant function has an inverse function.

c. Use the graph of the restricted secant function to graph $y = \sec^{-1} x$.

74. **a.** Graph the restricted contangent function, $y = \cot x$, by restricting x to the interval $(0, \pi)$.

b. Use the horizontal line test to explain why the restricted cotangent function has an inverse function.

c. Use the graph of the restricted cotangent function to graph $y = \cot^{-1} x$.

Practice Plus

The graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ are shown in **Table 4.10** *on page 557. In Exercises 75–84, use transformations (vertical shifts, horizontal shifts, reflections, stretching, or shrinking) of these graphs to graph each function. Then use interval notation to give the function's domain and range.*

75. $f(x) = \sin^{-1} x + \frac{\pi}{2}$

76. $f(x) = \cos^{-1} x + \frac{\pi}{2}$

77. $g(x) = \cos^{-1}(x + 1)$

78. $g(x) = \sin^{-1}(x + 1)$

79. $h(x) = -2\tan^{-1} x$

80. $h(x) = -3\tan^{-1} x$

81. $f(x) = \sin^{-1}(x - 2) - \frac{\pi}{2}$

82. $f(x) = \cos^{-1}(x - 2) - \frac{\pi}{2}$

83. $g(x) = \cos^{-1}\frac{x}{2}$

84. $g(x) = \sin^{-1}\frac{x}{2}$

In Exercises 85–92, determine the domain and the range of each function.

85. $f(x) = \sin(\sin^{-1} x)$

86. $f(x) = \cos(\cos^{-1} x)$

87. $f(x) = \cos^{-1}(\cos x)$

88. $f(x) = \sin^{-1}(\sin x)$

89. $f(x) = \sin^{-1}(\cos x)$

90. $f(x) = \cos^{-1}(\sin x)$

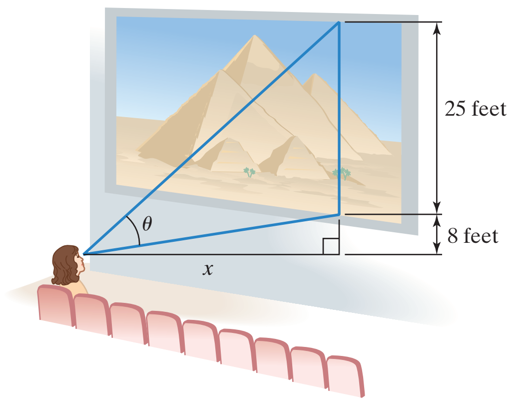
91. $f(x) = \sin^{-1} x + \cos^{-1} x$

92. $f(x) = \cos^{-1} x - \sin^{-1} x$

Application Exercises

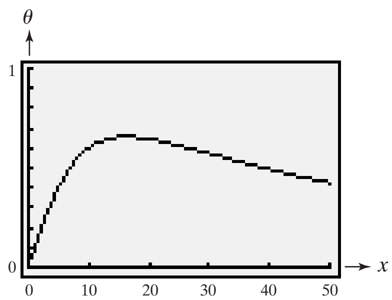
93. Your neighborhood movie theater has a 25-foot-high screen located 8 feet above your eye level. If you sit too close to the screen, your viewing angle is too small, resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit x feet back from the screen, your viewing angle, θ, is given by

$$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}.$$



Find the viewing angle, in radians, at distances of 5 feet, 10 feet, 15 feet, 20 feet, and 25 feet.

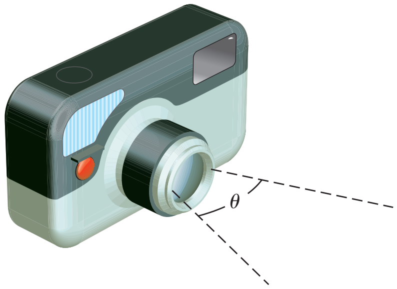
94. The function $\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$, described in Exercise 93, is graphed below in a [0, 50, 10] by [0, 1, 0.1] viewing rectangle. Use the graph to describe what happens to your viewing angle as you move farther back from the screen. How far back from the screen, to the nearest foot, should you sit to maximize your viewing angle? Verify this observation by finding the viewing angle one foot closer to the screen and one foot farther from the screen for this ideal viewing distance.



The formula

$$\theta = 2\tan^{-1}\frac{21.634}{x}$$

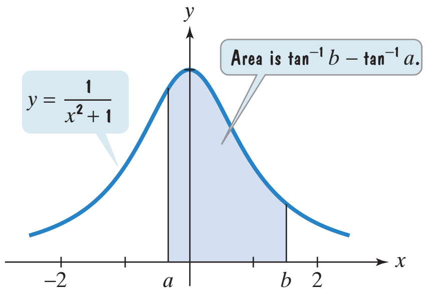
gives the viewing angle, θ, in radians, for a camera whose lens is x millimeters wide. Use this formula to solve Exercises 95–96.



95. Find the viewing angle, in radians and in degrees (to the nearest tenth of a degree), of a 28-millimeter lens.

96. Find the viewing angle, in radians and in degrees (to the nearest tenth of a degree), of a 300-millimeter telephoto lens.

For years, mathematicians were challenged by the following problem: What is the area of a region under a curve between two values of x? The problem was solved in the seventeenth century with the development of integral calculus. Using calculus, the area of the region under $y = \frac{1}{x^2+1}$, above the x-axis, and between $x = a$ and $x = b$ is $\tan^{-1} b - \tan^{-1} a$. Use this result, shown in the figure, to find the area of the region under $y = \frac{1}{x^2+1}$, above the x-axis, and between the values of a and b given in Exercises 97–98.



97. $a = 0$ and $b = 2$

98. $a = -2$ and $b = 1$

Writing in Mathematics

99. Explain why, without restrictions, no trigonometric function has an inverse function.

100. Describe the restriction on the sine function so that it has an inverse function.

101. How can the graph of $y = \sin^{-1} x$ be obtained from the graph of the restricted sine function?

102. Without drawing a graph, describe the behavior of the graph of $y = \sin^{-1} x$. Mention the function's domain and range in your description.

103. Describe the restriction on the cosine function so that it has an inverse function.

104. Without drawing a graph, describe the behavior of the graph of $y = \cos^{-1} x$. Mention the function's domain and range in your description.

105. Describe the restriction on the tangent function so that it has an inverse function.

106. Without drawing a graph, describe the behavior of the graph of $y = \tan^{-1} x$. Mention the function's domain and range in your description.

107. If $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$, is $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{5\pi}{6}$? Explain your answer.

108. Explain how a right triangle can be used to find the exact value of $\sec\left(\sin^{-1}\frac{4}{5}\right)$.

109. Find the height of the screen and the number of feet that it is located above eye level in your favorite movie theater. Modify the formula given in Exercise 93 so that it applies to your theater. Then describe where in the theater you should sit so that a movie creates the greatest visual impact.

Technology Exercises

In Exercises 110–113, graph each pair of functions in the same viewing rectangle. Use your knowledge of the domain and range for the inverse trigonometric function to select an appropriate viewing rectangle. How is the graph of the second equation in each exercise related to the graph of the first equation?

110. $y = \sin^{-1} x$ and $y = \sin^{-1} x + 2$

111. $y = \cos^{-1} x$ and $y = \cos^{-1}(x - 1)$

112. $y = \tan^{-1} x$ and $y = -2\tan^{-1} x$

113. $y = \sin^{-1} x$ and $y = \sin^{-1}(x + 2) + 1$

114. Graph $y = \tan^{-1} x$ and its two horizontal asymptotes in a $[-3, 3, 1]$ by $\left[-\pi, \pi, \frac{\pi}{2}\right]$ viewing rectangle. Then change the viewing rectangle to $[-50, 50, 5]$ by $\left[-\pi, \pi, \frac{\pi}{2}\right]$. What do you observe?

115. Graph $y = \sin^{-1} x + \cos^{-1} x$ in a $[-2, 2, 1]$ by $[0, 3, 1]$ viewing rectangle. What appears to be true about the sum of the inverse sine and inverse cosine for values between -1 and 1, inclusive?

Critical Thinking Exercises

Make Sense? *In Exercises 116–119, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

116. Because $y = \sin x$ has an inverse function if x is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, they should make restrictions easier to remember by also using $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as the restriction for $y = \cos x$.

117. Because $y = \sin x$ has an inverse function if x is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, they should make restrictions easier to remember by also using $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as the restriction for $y = \tan x$.

118. Although $\sin^{-1}\left(-\frac{1}{2}\right)$ is negative, $\cos^{-1}\left(-\frac{1}{2}\right)$ is positive.

119. I used $f^{-1}(f(x)) = x$ and concluded that

$$\sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}.$$

120. Solve $y = 2\sin^{-1}(x - 5)$ for x in terms of y.

121. Solve for x: $\quad 2\sin^{-1} x = \frac{\pi}{4}$.

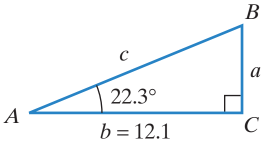
122. Prove that if $x > 0$, $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$.

123. Derive the formula for θ, your viewing angle at the movie theater, in Exercise 93. *Hint:* Use the figure shown and represent the acute angle on the left in the smaller right triangle by α. Find expressions for $\tan\alpha$ and $\tan(\alpha + \theta)$.

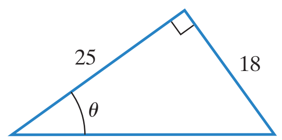
Preview Exercises

Exercises 124–126 will help you prepare for the material covered in the next section.

124. Use trigonometric functions to find a and c to two decimal places.



125. Find θ to the nearest tenth of a degree.



126. Determine the amplitude and period of $y = 10\cos\frac{\pi}{6}x$.