Group Exercise

106. Remembering the six sum and difference identities can be difficult. Did you have problems with some exercises because the identity you were using in your head turned out to be an incorrect formula? Are there easy ways to remember the six new identities presented in this section? Group members should address this question, considering one identity at a time. For each formula, list ways to make it easier to remember.

Preview Exercises

Exercises 107–109 will help you prepare for the material covered in the next section.

107. Give exact values for $\sin 30°$, $\cos 30°$, $\sin 60°$, and $\cos 60°$.

108. Use the appropriate values from Exercise 107 to answer each of the following.

a. Is $\sin 2 \cdot 30°$, or $\sin 60°$, equal to $2 \sin 30°$?

b. Is $\sin 2 \cdot 30°$, or $\sin 60°$, equal to $2 \sin 30° \cos 30°$?

109. Use appropriate values from Exercise 107 to answer each of the following.

a. Is $\cos(2 \cdot 30°)$, or $\cos 60°$, equal to $2 \cos 30°$?

b. Is $\cos(2 \cdot 30°)$, or $\cos 60°$, equal to $\cos^2 30° - \sin^2 30°$?

Section 5.3

Double-Angle, Power-Reducing, and Half-Angle Formulas

Objectives

1. Use the double-angle formulas.
2. Use the power-reducing formulas.
3. Use the half-angle formulas.

We have a long history of throwing things. Prior to 400 B.C., the Greeks competed in games that included discus throwing. In the seventeenth century, English soldiers organized cannonball-throwing competitions. In 1827, a Yale University student, disappointed over failing an exam, took out his frustrations at the passing of a collection plate in chapel. Seizing the monetary tray, he flung it in the direction of a large open space on campus. Yale students see this act of frustration as the origin of the Frisbee.

In this section, we develop other important classes of identities, called the double-angle, power-reducing, and half-angle formulas. We will see how one of these formulas can be used by athletes to increase throwing distance.

1 Use the double-angle formulas.

Double-Angle Formulas

A number of basic identities follow from the sum formulas for sine, cosine, and tangent. The first category of identities involves **double-angle formulas**.

Double-Angle Formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

To prove each of these formulas, we replace α and β by θ in the sum formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$.

Study Tip

The 2 that appears in each of the double-angle expressions cannot be pulled to the front and written as a coefficient.

Incorrect!

$$\cancel{\sin 2\theta = 2\sin\theta}$$
$$\cancel{\cos 2\theta = 2\cos\theta}$$
$$\cancel{\tan 2\theta = 2\tan\theta}$$

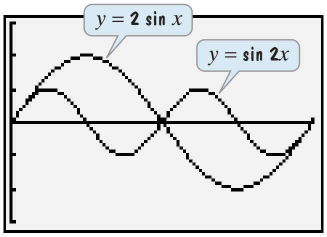
The figure shows that the graphs of

$$y = \sin 2x$$

and

$$y = 2\sin x$$

do not coincide: $\sin 2x \neq 2\sin x$.



$[0, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

- $\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

  We use $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

- $\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

  We use $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

- $\tan 2\theta = \tan(\theta + \theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

  We use $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

EXAMPLE 1 Using Double-Angle Formulas to Find Exact Values

If $\sin\theta = \frac{5}{13}$ and θ lies in quadrant II, find the exact value of each of the following:

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$.

Solution We begin with a sketch that illustrates

$$\sin\theta = \frac{5}{13} = \frac{y}{r}.$$

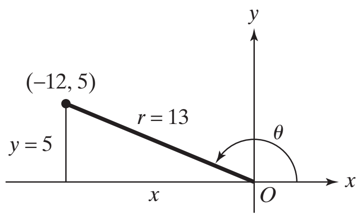


Figure 5.6 $\sin\theta = \frac{5}{13}$ and θ lies in quadrant II.

Figure 5.6 shows a quadrant II angle θ for which $\sin\theta = \frac{5}{13}$. We find x using $x^2 + y^2 = r^2$. Because θ lies in quadrant II, x is negative.

$$x^2 + 5^2 = 13^2 \qquad x^2 + y^2 = r^2$$
$$x^2 + 25 = 169 \qquad \text{Square 5 and 13, respectively.}$$
$$x^2 = 144 \qquad \text{Subtract 25 from both sides.}$$
$$x = -\sqrt{144} = -12 \qquad \text{If } x^2 = 144, \text{ then } x = \pm\sqrt{144} = \pm 12.$$

Choose $x = -\sqrt{144}$ because in quadrant II, x is negative.

Now we can use values for x, y, and r to find the required values. We will use $\cos\theta = \frac{x}{r} = -\frac{12}{13}$ and $\tan\theta = \frac{y}{x} = -\frac{5}{12}$. We were given $\sin\theta = \frac{5}{13}$.

a. $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = -\frac{120}{169}$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$

c. $$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\left(-\frac{5}{12}\right)}{1 - \left(-\frac{5}{12}\right)^2} = \frac{-\frac{5}{6}}{1 - \frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{119}{144}}$$
$$= \left(-\frac{5}{6}\right)\left(\frac{144}{119}\right) = -\frac{120}{119}$$

Check Point 1 If $\sin\theta = \frac{4}{5}$ and θ lies in quadrant II, find the exact value of each of the following:

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$.

EXAMPLE 2 Using the Double-Angle Formula for Tangent to Find an Exact Value

Find the exact value of $\dfrac{2\tan 15°}{1-\tan^2 15°}$.

Solution The given expression is the right side of the formula for $\tan 2\theta$ with $\theta = 15°$.

$$\underbrace{\frac{2\tan 15°}{1-\tan^2 15°}}_{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}} = \tan(2\cdot 15°) = \tan 30° = \frac{\sqrt{3}}{3}$$

Check Point 2 Find the exact value of $\cos^2 15° - \sin^2 15°$.

There are three forms of the double-angle formula for $\cos 2\theta$. The form we have seen involves both the cosine and the sine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

There are situations where it is more efficient to express $\cos 2\theta$ in terms of just one trigonometric function. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can write $\cos 2\theta = \cos^2\theta - \sin^2\theta$ in terms of the cosine only. We substitute $1 - \cos^2\theta$ for $\sin^2\theta$.

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta)\\ &= \cos^2\theta - 1 + \cos^2\theta = 2\cos^2\theta - 1\end{aligned}$$

We can also use a Pythagorean identity to write $\cos 2\theta$ in terms of sine only. We substitute $1 - \sin^2\theta$ for $\cos^2\theta$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

Three Forms of the Double-Angle Formula for cos 2θ

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

EXAMPLE 3 Verifying an Identity

Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Solution We begin by working with the left side. In order to obtain an expression for $\cos 3\theta$, we use the sum formula and write 3θ as $2\theta + \theta$.

$\cos 3\theta = \cos(2\theta + \theta)$ — Write 3θ as $2\theta + \theta$.

$= \underbrace{\cos 2\theta}_{2\cos^2\theta - 1}\cos\theta - \underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\sin\theta$ — $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$ — Substitute double-angle formulas. Because the right side of the given equation involves cosines only, use this form for $\cos 2\theta$.

$= 2\cos^3\theta - \cos\theta - 2\underbrace{\sin^2\theta}_{1-\cos^2\theta}\cos\theta$ — Multiply.

$$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

We've repeated the last step from the previous page.

$$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$$

To get cosines only, use $\sin^2\theta + \cos^2\theta = 1$ and substitute $1 - \cos^2\theta$ for $\sin^2\theta$.

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

Multiply.

$$= 4\cos^3\theta - 3\cos\theta$$

Simplify: $2\cos^3\theta + 2\cos^3\theta = 4\cos^3\theta$ and $-\cos\theta - 2\cos\theta = -3\cos\theta$.

We were required to verify $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. By working with the left side, $\cos 3\theta$, and expressing it in a form identical to the right side, we have verified the identity.

Check Point 3 Verify the identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

2 Use the power-reducing formulas.

Power-Reducing Formulas

The double-angle formulas are used to derive the **power-reducing formulas**:

Power-Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

We can prove the first two formulas in the box by working with two forms of the double-angle formula for $\cos 2\theta$.

This is the form with sine only. This is the form with cosine only.

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

Solve the formula on the left for $\sin^2\theta$. Solve the formula on the right for $\cos^2\theta$.

$$2\sin^2\theta = 1 - \cos 2\theta \qquad 2\cos^2\theta = 1 + \cos 2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

Divide both sides of each equation by 2.

These are the first two formulas in the box. The third formula in the box is proved by writing the tangent as the quotient of the sine and the cosine.

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1 - \cos 2\theta}{2}}{\frac{1 + \cos 2\theta}{2}} = \frac{1 - \cos 2\theta}{\cancel{2}_{1}} \cdot \frac{\cancel{2}^{1}}{1 + \cos 2\theta} = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Power-reducing formulas are quite useful in calculus. By reducing the power of trigonometric functions, calculus can better explore the relationship between a function and how it is changing at every single instant in time.

EXAMPLE 4 **Reducing the Power of a Trigonometric Function**

Write an equivalent expression for $\cos^4 x$ that does not contain powers of trigonometric functions greater than 1.

Solution We will apply the formula for $\cos^2\theta$ twice.

$$\cos^4 x = (\cos^2 x)^2$$

$$= \left(\frac{1 + \cos 2x}{2}\right)^2 \qquad \text{Use } \cos^2\theta = \frac{1+\cos 2\theta}{2} \text{ with } \theta = x.$$

$$= \frac{1 + 2\cos 2x + \cos^2 2x}{4} \qquad \text{Square the numerator: } (A+B)^2 = A^2 + 2AB + B^2. \text{ Square the denominator.}$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \qquad \text{Divide each term in the numerator by 4.}$$

We can reduce the power of $\cos^2 2x$ using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ with $\theta = 2x$.

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left[\frac{1 + \cos 2(2x)}{2}\right] \qquad \text{Use the power-reducing formula for } \cos^2\theta \text{ with } \theta = 2x.$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x) \qquad \text{Multiply.}$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x \qquad \text{Distribute } \tfrac{1}{8} \text{ throughout parentheses.}$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \qquad \text{Simplify: } \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{2}{8} + \tfrac{1}{8} = \tfrac{3}{8}.$$

Thus, $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$. The expression for $\cos^4 x$ does not contain powers of trigonometric functions greater than 1.

Check Point 4 Write an equivalent expression for $\sin^4 x$ that does not contain powers of trigonometric functions greater than 1.

3 Use the half-angle formulas.

Half-Angle Formulas

Useful equivalent forms of the power-reducing formulas can be obtained by replacing θ with $\frac{\alpha}{2}$. Then solve for the trigonometric function on the left sides of the equations. The resulting identities are called the **half-angle formulas**:

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}$$

The $\pm$ symbol in each formula does not mean that there are two possible values for each function. Instead, the $\pm$ indicates that you must determine the sign of the trigonometric function, + or −, based on the quadrant in which the half-angle $\frac{\alpha}{2}$ lies.

Study Tip

The $\frac{1}{2}$ that appears in each of the half-angle formulas cannot be pulled to the front and written as a coefficient.

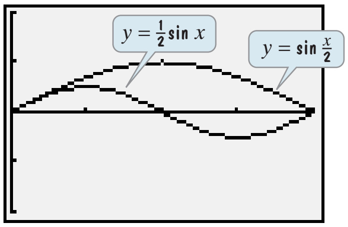
Incorrect!

$$\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta$$

$$\cos\frac{\theta}{2} = \frac{1}{2}\cos\theta$$

$$\tan\frac{\theta}{2} = \frac{1}{2}\tan\theta$$

The figure shows that the graphs of $y = \sin\frac{x}{2}$ and $y = \frac{1}{2}\sin x$ do not coincide: $\sin\frac{x}{2} \neq \frac{1}{2}\sin x$.



$[0, 2\pi, \frac{\pi}{2}]$ by $[-2, 2, 1]$

If we know the exact value for the cosine of an angle, we can use the half-angle formulas to find exact values of sine, cosine, and tangent for half of that angle. For example, we know that $\cos 225° = -\frac{\sqrt{2}}{2}$. In the next example, we find the exact value of the cosine of half of 225°, or cos 112.5°.

EXAMPLE 5 Using a Half-Angle Formula to Find an Exact Value

Find the exact value of cos 112.5°.

Solution Because $112.5° = \frac{225°}{2}$, we use the half-angle formula for $\cos\frac{\alpha}{2}$ with $\alpha = 225°$. What sign should we use when we apply the formula? Because 112.5° lies in quadrant II, where only the sine and cosecant are positive, $\cos 112.5° < 0$. Thus, we use the $-$ sign in the half-angle formula.

$$\cos 112.5° = \cos\frac{225°}{2}$$

$$= -\sqrt{\frac{1+\cos 225°}{2}} \qquad \text{Use } \cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} \text{ with } \alpha = 225°.$$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}} \qquad \cos 225° = -\frac{\sqrt{2}}{2}$$

$$= -\sqrt{\frac{2-\sqrt{2}}{4}} \qquad \text{Multiply the radicand by } \tfrac{2}{2}\text{: } \frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}\cdot\frac{2}{2} = \frac{2-\sqrt{2}}{4}.$$

$$= -\frac{\sqrt{2-\sqrt{2}}}{2} \qquad \text{Simplify: } \sqrt{4} = 2.$$

Discovery

Use your calculator to find approximations for

$$-\frac{\sqrt{2-\sqrt{2}}}{2}$$

and cos 112.5°. What do you observe?

Study Tip

Keep in mind as you work with the half-angle formulas that the sign *outside* the radical is determined by the half angle $\frac{\alpha}{2}$. By contrast, the sign of $\cos\alpha$, which appears *under* the radical, is determined by the full angle α.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

The sign is determined by the quadrant of $\frac{\alpha}{2}$.

The sign of $\cos\alpha$ is determined by the quadrant of α.

Check Point 5 Use $\cos 210° = -\frac{\sqrt{3}}{2}$ to find the exact value of cos 105°.

There are alternate formulas for $\tan\frac{\alpha}{2}$ that do not require us to determine what sign to use when applying the formula. These formulas are logically connected to the identities in Example 6 and Check Point 6.

EXAMPLE 6 Verifying an Identity

Verify the identity: $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$.

Solution We work with the right side.

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

The form $\cos 2\theta = 1 - 2\sin^2\theta$ is used because it produces only one term in the numerator. Use the double-angle formula for sine in the denominator.

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

Simplify the numerator.

$$= \frac{\sin\theta}{\cos\theta}$$

Divide the numerator and denominator by $2\sin\theta$.

$$= \tan\theta$$

Use a quotient identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

The right side simplifies to $\tan\theta$, the expression on the left side. Thus, the identity is verified.

Check Point 6 Verify the identity: $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$.

Half-angle formulas for $\tan\dfrac{\alpha}{2}$ can be obtained using the identities in Example 6 and Check Point 6:

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} \quad \text{and} \quad \tan\theta = \frac{\sin 2\theta}{1+\cos 2\theta}.$$

Do you see how to do this? Replace each occurrence of θ with $\dfrac{\alpha}{2}$. This results in the following identities:

Half-Angle Formulas for Tangent

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

EXAMPLE 7 Verifying an Identity

Verify the identity: $\tan\dfrac{\alpha}{2} = \csc\alpha - \cot\alpha$.

Solution We begin with the right side.

$$\csc\alpha - \cot\alpha = \underbrace{\frac{1}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha}}_{\text{Express functions in terms of sines and cosines.}} = \underbrace{\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}}_{\text{This is the first of the two half-angle formulas in the preceding box.}}$$

We worked with the right side and arrived at the left side. Thus, the identity is verified.

Check Point 7 Verify the identity: $\tan\dfrac{\alpha}{2} = \dfrac{\sec\alpha}{\sec\alpha\csc\alpha + \csc\alpha}$.

We conclude with a summary of the principal trigonometric identities developed in this section and the previous section. The fundamental identities can be found in the box on page 586.

Principal Trigonometric Identities

Sum and Difference Formulas

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$$

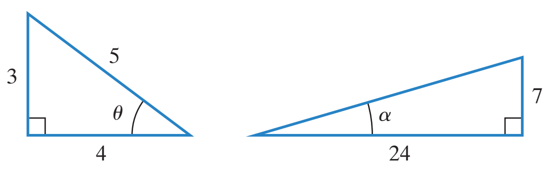
Study Tip

To help remember the correct sign in the numerator in the first two power-reducing formulas and the first two half-angle formulas, remember *sinus-minus*—the sine is minus.

Exercise Set 5.3

Practice Exercises

In Exercises 1–6, use the figures to find the exact value of each trigonometric function.



1. $\sin 2\theta$ **2.** $\cos 2\theta$ **3.** $\tan 2\theta$

4. $\sin 2\alpha$ **5.** $\cos 2\alpha$ **6.** $\tan 2\alpha$

In Exercises 7–14, use the given information to find the exact value of each of the following:

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$.

7. $\sin\theta = \frac{15}{17}$, θ lies in quadrant II.

8. $\sin\theta = \frac{12}{13}$, θ lies in quadrant II.

9. $\cos\theta = \frac{24}{25}$, θ lies in quadrant IV.

10. $\cos\theta = \frac{40}{41}$, θ lies in quadrant IV.

11. $\cot\theta = 2$, θ lies in quadrant III.

12. $\cot\theta = 3$, θ lies in quadrant III.

13. $\sin\theta = -\frac{9}{41}$, θ lies in quadrant III.

14. $\sin\theta = -\frac{2}{3}$, θ lies in quadrant III.

In Exercises 15–22, write each expression as the sine, cosine, or tangent of a double angle. Then find the exact value of the expression.

15. $2\sin 15^\circ \cos 15^\circ$

16. $2\sin 22.5^\circ \cos 22.5^\circ$

17. $\cos^2 75^\circ - \sin^2 75^\circ$

18. $\cos^2 105^\circ - \sin^2 105^\circ$

19. $2\cos^2\dfrac{\pi}{8} - 1$

20. $1 - 2\sin^2\dfrac{\pi}{12}$

21. $\dfrac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$

22. $\dfrac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$

In Exercises 23–34, verify each identity.

23. $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

24. $\sin 2\theta = \dfrac{2\cot\theta}{1 + \cot^2\theta}$

25. $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

26. $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$

27. $\sin^2 x + \cos 2x = \cos^2 x$

28. $1 - \tan^2 x = \dfrac{\cos 2x}{\cos^2 x}$

29. $\cot x = \dfrac{\sin 2x}{1 - \cos 2x}$

30. $\cot x = \dfrac{1 + \cos 2x}{\sin 2x}$

31. $\sin 2t - \tan t = \tan t \cos 2t$

32. $\sin 2t - \cot t = -\cot t \cos 2t$

33. $\sin 4t = 4\sin t\cos^3 t - 4\sin^3 t\cos t$

34. $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$

In Exercises 35–38, use the power-reducing formulas to rewrite each expression as an equivalent expression that does not contain powers of trigonometric functions greater than 1.

35. $6\sin^4 x$

36. $10\cos^4 x$

37. $\sin^2 x\cos^2 x$

38. $8\sin^2 x\cos^2 x$

In Exercises 39–46, use a half-angle formula to find the exact value of each expression.

39. $\sin 15^\circ$

40. $\cos 22.5^\circ$

41. $\cos 157.5^\circ$

42. $\sin 105^\circ$

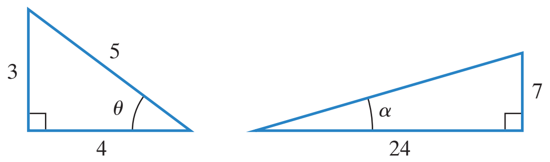
43. $\tan 75^\circ$

44. $\tan 112.5^\circ$

45. $\tan\dfrac{7\pi}{8}$

46. $\tan\dfrac{3\pi}{8}$

In Exercises 47–54, use the figures to find the exact value of each trigonometric function.



47. $\sin\dfrac{\theta}{2}$

48. $\cos\dfrac{\theta}{2}$

49. $\tan\dfrac{\theta}{2}$

50. $\sin\dfrac{\alpha}{2}$

51. $\cos\dfrac{\alpha}{2}$

52. $\tan\dfrac{\alpha}{2}$

53. $2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

54. $2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}$

In Exercises 55–58, use the given information to find the exact value of each of the following:

a. $\sin\dfrac{\alpha}{2}$ **b.** $\cos\dfrac{\alpha}{2}$ **c.** $\tan\dfrac{\alpha}{2}$.

55. $\tan\alpha = \frac{4}{3}$, $180^\circ < \alpha < 270^\circ$

56. $\tan\alpha = \frac{8}{15}$, $180^\circ < \alpha < 270^\circ$

57. $\sec\alpha = -\frac{13}{5}$, $\frac{\pi}{2} < \alpha < \pi$

58. $\sec\alpha = -3$, $\frac{\pi}{2} < \alpha < \pi$

In Exercises 59–68, verify each identity.

59. $\sin^2\dfrac{\theta}{2} = \dfrac{\sec\theta - 1}{2\sec\theta}$

60. $\sin^2\dfrac{\theta}{2} = \dfrac{\csc\theta - \cot\theta}{2\csc\theta}$

61. $\cos^2\dfrac{\theta}{2} = \dfrac{\sin\theta + \tan\theta}{2\tan\theta}$

62. $\cos^2\dfrac{\theta}{2} = \dfrac{\sec\theta + 1}{2\sec\theta}$

63. $\tan\dfrac{\alpha}{2} = \dfrac{\tan\alpha}{\sec\alpha + 1}$

64. $2\tan\dfrac{\alpha}{2} = \dfrac{\sin^2\alpha + 1 - \cos^2\alpha}{\sin\alpha(1 + \cos\alpha)}$

65. $\cot\dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$

66. $\cot\dfrac{x}{2} = \dfrac{1 + \cos x}{\sin x}$

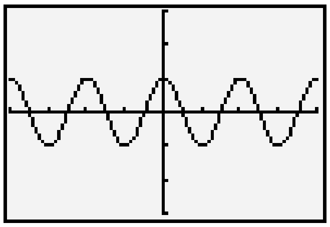
67. $\tan\dfrac{x}{2} + \cot\dfrac{x}{2} = 2\csc x$

68. $\tan\dfrac{x}{2} - \cot\dfrac{x}{2} = -2\cot x$

Practice Plus

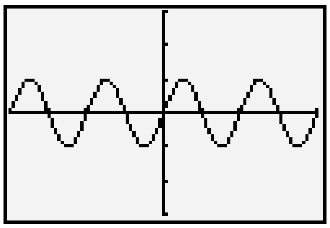
In Exercises 69–78, half of an identity and the graph of this half are given. Use the graph to make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

69. $\dfrac{\cot x - \tan x}{\cot x + \tan x} = ?$



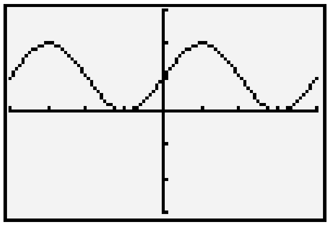
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

70. $\dfrac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = ?$



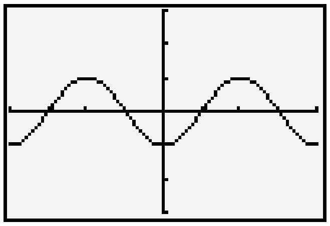
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

71. $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = \ ?$



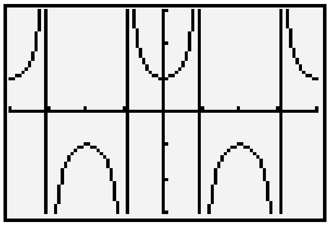
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

72. $\sin^2\frac{x}{2} - \cos^2\frac{x}{2} = \ ?$



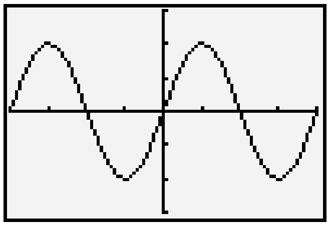
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

73. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \ ?$



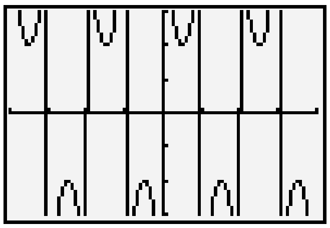
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

74. $\sin 2x \sec x = \ ?$



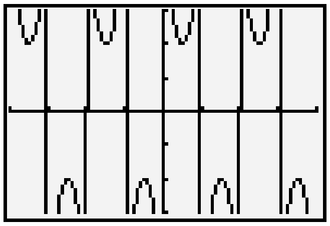
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

75. $\dfrac{\csc^2 x}{\cot x} = \ ?$



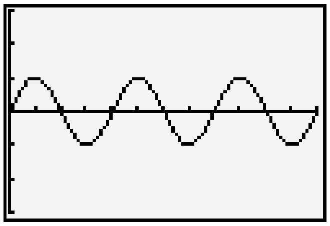
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

76. $\tan x + \cot x = \ ?$



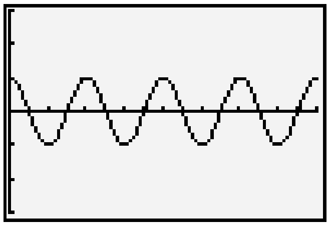
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-3, 3, 1]$

77. $\sin x(4\cos^2 x - 1) = \ ?$



$[0, 2\pi, \frac{\pi}{6}]$ by $[-3, 3, 1]$

78. $1 - 8\sin^2 x \cos^2 x = \ ?$



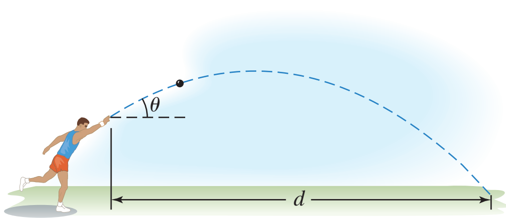
$[0, 2\pi, \frac{\pi}{8}]$ by $[-3, 3, 1]$

Application Exercises

79. Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ in the figure shown. The distance, d, in feet, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta,$$

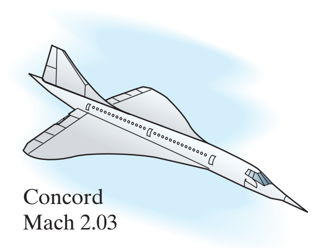
in which v_0 is the initial speed of the object thrown, in feet per second, and θ is the angle, in degrees, at which the object leaves the hand.



a. Use an identity to express the formula so that it contains the sine function only.

b. Use your formula from part (a) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

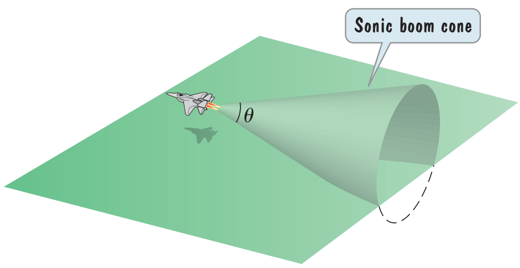
Use this information to solve Exercises 80–81: The speed of a supersonic aircraft is usually represented by a Mach number, named after Austrian physicist Ernst Mach (1838–1916). A Mach number is the speed of the aircraft, in miles per hour, divided by the speed of sound, approximately 740 miles per hour. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.



Concord
Mach 2.03

SR-71 Blackbird
Mach 3.3

If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ, shown in the figure.



The relationship between the cone's vertex angle, θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}.$$

80. If $\theta = \frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as a decimal to the nearest tenth.

81. If $\theta = \frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as a decimal to the nearest tenth.

Writing in Mathematics

In Exercises 82–89, use words to describe the formula for:

82. the sine of double an angle.

83. the cosine of double an angle. (Describe one of the three formulas.)

84. the tangent of double an angle.

85. the power-reducing formula for the sine squared of an angle.

86. the power-reducing formula for the cosine squared of an angle.

87. the sine of half an angle.

88. the cosine of half an angle.

89. the tangent of half an angle. (Describe one of the two formulas that does not involve a square root.)

90. Explain how the double-angle formulas are derived.

91. How can there be three forms of the double-angle formula for $\cos 2\theta$?

92. Without showing algebraic details, describe in words how to reduce the power of $\cos^4 x$.

93. Describe one or more of the techniques you use to help remember the identities in the box on page 614.

94. Your friend is about to compete as a shot-putter in a college field event. Using Exercise 79(b), write a short description to your friend on how to achieve the best distance possible in the throwing event.

Technology Exercises

In Exercises 95–98, graph each side of the equation in the same viewing rectangle. If the graphs appear to coincide, verify that the equation is an identity. If the graphs do not appear to coincide, find a value of x for which both sides are defined but not equal.

95. $3 - 6\sin^2 x = 3\cos 2x$

96. $4\cos^2\frac{x}{2} = 2 + 2\cos x$

97. $\sin\frac{x}{2} = \frac{1}{2}\sin x$

98. $\cos\frac{x}{2} = \frac{1}{2}\cos x$

In Exercises 99–101, graph each equation in a $\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-3, 3, 1]$ viewing rectangle. Then **a.** *Describe the graph using another equation, and* **b.** *Verify that the two equations are equivalent.*

99. $y = \dfrac{1 - 2\cos 2x}{2\sin x - 1}$

100. $y = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

101. $y = \csc x - \cot x$

Critical Thinking Exercises

Make Sense? *In Exercises 102–105, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

102. The double-angle identities are derived from the sum identities by adding an angle to itself.

103. I simplified a double-angle trigonometric expression by pulling 2 to the front and treating it as a coefficient.

104. When using the half-angle formulas for trigonometric functions of $\frac{\alpha}{2}$, I determine the sign based on the quadrant in which α lies.

105. I used a half-angle formula to find the exact value of $\cos 100°$.

106. Verify the identity:

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)\left(1 - \frac{\sin 2x}{2}\right).$$

In Exercises 107–110, find the exact value of each expression. Do not use a calculator.

107. $\sin\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right)$

108. $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

109. $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$

110. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$

111. Use a right triangle to write $\sin(2\sin^{-1}x)$ as an algebraic expression. Assume that x is positive and in the domain of the given inverse trigonometric function.

112. Use the power-reducing formulas to rewrite $\sin^6 x$ as an equivalent expression that does not contain powers of trigonometric functions greater than 1.

Preview Exercises

Exercises 113–115 will help you prepare for the material covered in the next section. In each exercise, use exact values of trigonometric functions to show that the statement is true. Notice that each statement expresses the product of sines and/or cosines as a sum or a difference.

113. $\sin 60^\circ \sin 30^\circ = \frac{1}{2}[\cos(60^\circ - 30^\circ) - \cos(60^\circ + 30^\circ)]$

114. $\cos\frac{\pi}{2}\cos\frac{\pi}{3} = \frac{1}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)\right]$

115. $\sin\pi\cos\frac{\pi}{2} = \frac{1}{2}\left[\sin\left(\pi + \frac{\pi}{2}\right) + \sin\left(\pi - \frac{\pi}{2}\right)\right]$

Chapter 5 Mid-Chapter Check Point

What you Know: Verifying an identity means showing that the expressions on each side are identical. Like solving puzzles, the process can be intriguing because there are sometimes several "best" ways to proceed. We presented some guidelines to help you get started (see page 593). We used fundamental trigonometric identities (see page 586), as well as sum and difference formulas, double-angle formulas, power-reducing formulas, and half-angle formulas (see page 614) to verify identities. We also used these formulas to find exact values of trigonometric functions.

Study Tip

Make copies of the boxes on pages 586 and 614 that contain the essential trigonometric identities. Mount these boxes on cardstock and add this reference sheet to the one you prepared for Chapter 4. (If you didn't prepare a reference sheet for Chapter 4, it's not too late: See the study tip on page 580.)

In Exercises 1–18, verify each identity.

1. $\cos x(\tan x + \cot x) = \csc x$

2. $\dfrac{\sin(x + \pi)}{\cos\left(x + \frac{3\pi}{2}\right)} = \tan^2 x - \sec^2 x$

3. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

4. $\dfrac{\sin t - 1}{\cos t} = \dfrac{\cos t - \cot t}{\cos t \cot t}$

5. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$

6. $\sin\theta\cos\theta + \cos^2\theta = \dfrac{\cos\theta(1 + \cot\theta)}{\csc\theta}$

7. $\dfrac{\sin x}{\tan x} + \dfrac{\cos x}{\cot x} = \sin x + \cos x$

8. $\sin^2\dfrac{t}{2} = \dfrac{\tan t - \sin t}{2\tan t}$

9. $\sin\alpha\cos\beta = \dfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

10. $\dfrac{1 + \csc x}{\sec x} - \cot x = \cos x$

11. $\dfrac{\cot x - 1}{\cot x + 1} = \dfrac{1 - \tan x}{1 + \tan x}$

12. $2\sin^3\theta\cos\theta + 2\sin\theta\cos^3\theta = \sin 2\theta$

13. $\dfrac{\sin t + \cos t}{\sec t + \csc t} = \dfrac{\sin t}{\sec t}$

14. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$

15. $\tan(\alpha + \beta)\tan(\alpha - \beta) = \dfrac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\tan^2\beta}$

16. $\csc\theta + \cot\theta = \dfrac{\sin\theta}{1 - \cos\theta}$

17. $\dfrac{1}{\csc 2x} = \dfrac{2\tan x}{1 + \tan^2 x}$

18. $\dfrac{\sec t - 1}{t\sec t} = \dfrac{1 - \cos t}{t}$

Use the following conditions to solve Exercises 19–22:

$$\sin\alpha = \frac{3}{5}, \quad \frac{\pi}{2} < \alpha < \pi$$
$$\cos\beta = -\frac{12}{13}, \quad \pi < \beta < \frac{3\pi}{2}.$$

Find the exact value of each of the following.

19. $\cos(\alpha - \beta)$

20. $\tan(\alpha + \beta)$

21. $\sin 2\alpha$

22. $\cos\dfrac{\beta}{2}$

In Exercises 23–26, find the exact value of each expression. Do not use a calculator.

23. $\sin\left(\dfrac{3\pi}{4} + \dfrac{5\pi}{6}\right)$

24. $\cos^2 15^\circ - \sin^2 15^\circ$

25. $\cos\dfrac{5\pi}{12}\cos\dfrac{\pi}{12} + \sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12}$

26. $\tan 22.5^\circ$