Section 5.4

Product-to-Sum and Sum-to-Product Formulas

Objectives

1. Use the product-to-sum formulas.
2. Use the sum-to-product formulas.

James K. Polk
Born November 2, 1795

Warren G. Harding
Born November 2, 1865

Of the 43 U.S. presidents, two share a birthday (same month and day). The probability of two or more people in a group sharing a birthday rises sharply as the group's size increases. Above 50 people, the probability approaches certainty. (You can verify the mathematics of this surprising result by studying Sections 10.6 and 10.7, and working Exercise 73 in Exercise Set 10.7.) So, come November 2, we salute Presidents Polk and Harding with

112, 163-, 112, 196-, 110, 8521-, 008, 121-.

Were you aware that each button on your touch-tone phone produces a unique sound? If we treat the commas as pauses and the hyphens as held notes, this sequence of numbers is *Happy Birthday* on a touch-tone phone.

Although *Happy Birthday* isn't Mozart or Sondheim, it is sinusoidal. Each of its touch-tone musical sounds can be described by the sum of two sine functions or the product of sines and cosines. In this section, we develop identities that enable us to use both descriptions. They are called the product-to-sum and sum-to-product formulas.

1 Use the product-to-sum formulas.

The Product-to-Sum Formulas

How do we write the products of sines and/or cosines as sums or differences? We use the following identities, which are called **product-to-sum formulas**:

Study Tip

You may not need to memorize the formulas in this section. When you need them, you can either refer to one of the two boxes in the section or perhaps even derive them using the methods shown.

Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Although these formulas are difficult to remember, they are fairly easy to derive. For example, let's derive the first identity in the box,

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

We begin with the difference and sum formulas for the cosine, and subtract the second identity from the first:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ -[\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta] \\ \hline \cos(\alpha - \beta) - \cos(\alpha + \beta) &= \quad 0 \quad + 2\sin\alpha\sin\beta. \end{aligned}$$

Subtract terms on the left side.

Subtract terms on the right side: $\cos\alpha\cos\beta - \cos\alpha\cos\beta = 0$.

Subtract terms on the right side: $\sin\alpha\sin\beta - (-\sin\alpha\sin\beta) = 2\sin\alpha\sin\beta$.

Now we use this result to derive the product-to-sum formula for $\sin\alpha\sin\beta$.

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \qquad \text{Reverse the sides in the preceding equation.}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \text{Multiply each side by } \tfrac{1}{2}.$$

This last equation is the desired formula. Likewise, we can derive the product-to-sum formula for cosine, $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$. As we did for the previous derivation, begin with the difference and sum formulas for cosine. However, we *add* the formulas rather than subtracting them. Reversing both sides of this result and multiplying each side by $\frac{1}{2}$ produces the formula for $\cos\alpha\cos\beta$. The last two product-to-sum formulas, $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ and $\cos\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$, are derived using the sum and difference formulas for sine in a similar manner.

Technology

Graphic Connections

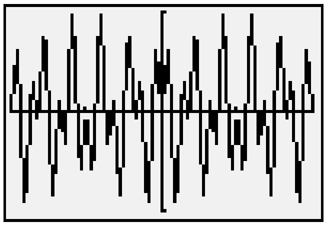
The graphs of

$$y = \sin 8x \sin 3x$$

and

$$y = \tfrac{1}{2}(\cos 5x - \cos 11x)$$

are shown in a $\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-1, 1, 1]$ viewing rectangle. The graphs coincide. This supports our algebraic work in Example 1(a).



EXAMPLE 1 Using the Product-to-Sum Formulas

Express each of the following products as a sum or difference:

a. $\sin 8x \sin 3x$ **b.** $\sin 4x \cos x$.

Solution The product-to-sum formula that we are using is shown in each of the voice balloons.

a. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$$\sin 8x \sin 3x = \frac{1}{2}[\cos(8x - 3x) - \cos(8x + 3x)] = \frac{1}{2}(\cos 5x - \cos 11x)$$

b. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$$\sin 4x \cos x = \frac{1}{2}[\sin(4x + x) + \sin(4x - x)] = \frac{1}{2}(\sin 5x + \sin 3x)$$

Check Point 1 Express each of the following products as a sum or difference:

a. $\sin 5x \sin 2x$ **b.** $\cos 7x \cos x$.

2 Use the sum-to-product formulas.

The Sum-to-Product Formulas

How do we write the sum or difference of sines and/or cosines as products? We use the following identities, which are called the **sum-to-product formulas**:

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}$$

We verify these formulas using the product-to-sum formulas. Let's verify the first sum-to-product formula

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

We start with the right side of the formula, the side with the product. We can apply the product-to-sum formula for $\sin \alpha \cos \beta$ to this expression. By doing so, we obtain the left side of the formula, $\sin \alpha + \sin \beta$. Here's how:

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = 2 \cdot \frac{1}{2}\left[\sin\left(\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2}\right) + \sin\left(\frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2}\right)\right]$$

$$= \sin\left(\frac{\alpha + \beta + \alpha - \beta}{2}\right) + \sin\left(\frac{\alpha + \beta - \alpha + \beta}{2}\right)$$

$$= \sin \frac{2\alpha}{2} + \sin \frac{2\beta}{2} = \sin \alpha + \sin \beta.$$

The three other sum-to-product formulas in the box at the bottom of the previous page are verified in a similar manner. Start with the right side and obtain the left side using an appropriate product-to-sum formula.

EXAMPLE 2 Using the Sum-to-Product Formulas

Express each sum or difference as a product:

a. $\sin 9x + \sin 5x$ **b.** $\cos 4x - \cos 3x$.

Solution The sum-to-product formula that we are using is shown in each of the voice balloons.

a.

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin 9x + \sin 5x = 2 \sin \frac{9x + 5x}{2} \cos \frac{9x - 5x}{2}$$

$$= 2 \sin \frac{14x}{2} \cos \frac{4x}{2}$$

$$= 2 \sin 7x \cos 2x$$

b.

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos 4x - \cos 3x = -2 \sin \frac{4x + 3x}{2} \sin \frac{4x - 3x}{2}$$

$$= -2 \sin \frac{7x}{2} \sin \frac{x}{2}$$

Check Point 2 Express each sum as a product:

a. $\sin 7x + \sin 3x$ **b.** $\cos 3x + \cos 2x$.

Some identities contain a fraction on one side with sums and differences of sines and/or cosines. Applying the sum-to-product formulas in the numerator and the denominator is often helpful in verifying these identities.

EXAMPLE 3 Using Sum-to-Product Formulas to Verify an Identity

Verify the identity: $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x.$

Solution Because the left side of $\frac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$ is more complicated, we will work with it. We use sum-to-product formulas for the numerator and the denominator of the fraction on this side.

$$\frac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x}$$

$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$

$$= \frac{-2 \sin \frac{3x + 5x}{2} \sin \frac{3x - 5x}{2}}{\sin 3x + \sin 5x}$$

$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$

$$= \frac{-2 \sin \frac{3x + 5x}{2} \sin \frac{3x - 5x}{2}}{2 \sin \frac{3x + 5x}{2} \cos \frac{3x - 5x}{2}}$$

$$= \frac{-2 \sin \frac{8x}{2} \sin\left(\frac{-2x}{2}\right)}{2 \sin \frac{8x}{2} \cos\left(\frac{-2x}{2}\right)}$$ Perform the indicated additions and subtractions.

$$= \frac{-2 \sin 4x \sin(-x)}{2 \sin 4x \cos(-x)}$$ Simplify.

$$= \frac{-(-\sin x)}{\cos x}$$ The sine function is odd: $\sin(-x) = -\sin x$. The cosine function is even: $\cos(-x) = \cos x$.

$$= \frac{\sin x}{\cos x}$$ Simplify.

$$= \tan x$$ Apply a quotient identity: $\tan x = \frac{\sin x}{\cos x}$.

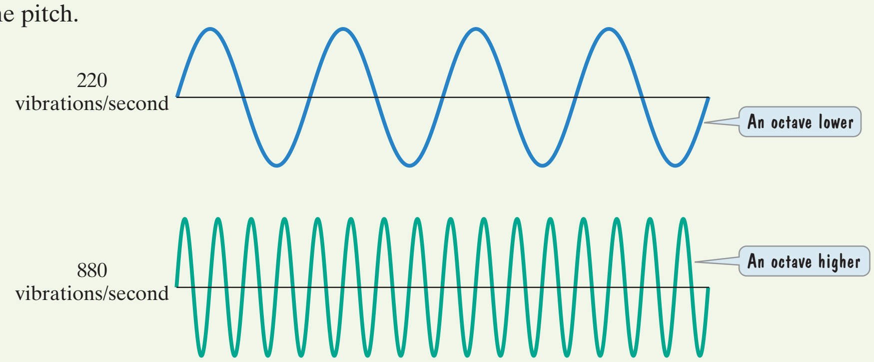
We were required to verify $\frac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$. We worked with the left side and arrived at the right side, $\tan x$. Thus, the identity is verified.

Check Point 3 Verify the identity: $\frac{\cos 3x - \cos x}{\sin 3x + \sin x} = -\tan x$.

Sinusoidal Sounds

Music is all around us. A mere snippet of a song from the past can trigger vivid memories, inducing emotions ranging from unabashed joy to deep sorrow. Trigonometric functions can explain how sound travels from its source and describe its pitch, loudness, and quality. Still unexplained is the remarkable influence music has on the brain, including the deepest question of all: Why do we appreciate music?

When a note is played, it disturbs nearby air molecules, creating regions of higher-than-normal pressure and regions of lower-than-normal pressure. If we graph pressure, y, versus time, t, we get a sine wave that represents the note. The frequency of the sine wave is the number of high-low disturbances, or vibrations, per second. The greater the frequency, the higher the pitch; the lesser the frequency, the lower the pitch.



The amplitude of a note's sine wave is related to its loudness. The amplitude for the two sine waves shown above is the same. Thus, the notes have the same loudness, although they differ in pitch. The greater the amplitude, the louder the sound; the lesser the amplitude, the softer the sound. The amplitude and frequency are characteristic of every note—and thus of its graph—until the note dissipates.

Exercise Set 5.4

Practice Exercises

Because you may not be required to memorize the identities in this section, it's often tempting to pay no attention to them at all! Exercises 1–4 are provided to familiarize you with what these identities do. Fill in each blank using the word sum, difference, product, *or* quotient.

1. The formula
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
can be used to change a _______ of two sines into the _______ of two cosine expressions.

2. The formula
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
can be used to change a _______ of two cosines into the _______ of two cosine expressions.

3. The formula
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$
can be used to change a _______ of a sine and a cosine into the _______ of two sine expressions.

4. The formula
$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$
can be used to change a _______ of a cosine and a sine into the _______ of two sine expressions.

Now that you've familiarized yourself with the formulas, in Exercises 5–12, use the appropriate formula to express each product as a sum or difference.

5. $\sin 6x\sin 2x$
6. $\sin 8x\sin 4x$
7. $\cos 7x\cos 3x$
8. $\cos 9x\cos 2x$
9. $\sin x\cos 2x$
10. $\sin 2x\cos 3x$
11. $\cos\frac{3x}{2}\sin\frac{x}{2}$
12. $\cos\frac{5x}{2}\sin\frac{x}{2}$

Exercises 13–16 are provided to familiarize you with the second set of identities presented in this section. Fill in each blank using the word sum, difference, product, *or* quotient.

13. The formula
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
can be used to change a _______ of two sines into the _______ of a sine and a cosine expression.

14. The formula
$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$
can be used to change a _______ of two sines into the _______ of a sine and a cosine expression.

15. The formula
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
can be used to change a _______ of two cosines into the _______ of two cosine expressions.

16. The formula
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
can be used to change a _______ of two cosines into the _______ of two sine expressions.

Now that you've familiarized yourself with the second set of formulas presented in this section, in Exercises 17–30, express each sum or difference as a product. If possible, find this product's exact value.

17. $\sin 6x + \sin 2x$
18. $\sin 8x + \sin 2x$
19. $\sin 7x - \sin 3x$
20. $\sin 11x - \sin 5x$
21. $\cos 4x + \cos 2x$
22. $\cos 9x - \cos 7x$
23. $\sin x + \sin 2x$
24. $\sin x - \sin 2x$
25. $\cos\frac{3x}{2} + \cos\frac{x}{2}$
26. $\sin\frac{3x}{2} + \sin\frac{x}{2}$
27. $\sin 75° + \sin 15°$
28. $\cos 75° - \cos 15°$
29. $\sin\frac{\pi}{12} - \sin\frac{5\pi}{12}$
30. $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12}$

In Exercises 31–38, verify each identity.

31. $\frac{\sin 3x - \sin x}{\cos 3x - \cos x} = -\cot 2x$
32. $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$
33. $\frac{\sin 2x + \sin 4x}{\cos 2x + \cos 4x} = \tan 3x$
34. $\frac{\cos 4x - \cos 2x}{\sin 2x - \sin 4x} = \tan 3x$
35. $\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\frac{x-y}{2}\cot\frac{x+y}{2}$
36. $\frac{\sin x + \sin y}{\sin x - \sin y} = \tan\frac{x+y}{2}\cot\frac{x-y}{2}$
37. $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\frac{x+y}{2}$
38. $\frac{\sin x - \sin y}{\cos x - \cos y} = -\cot\frac{x+y}{2}$

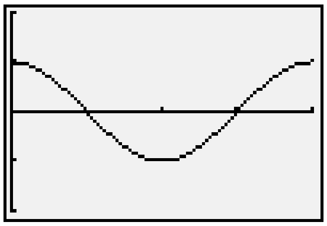
Practice Plus

In Exercises 39–44, the graph with the given equation is shown in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ *by* $[-2, 2, 1]$ *viewing rectangle.*

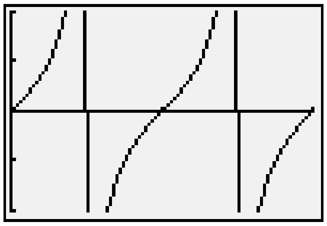
a. *Describe the graph using another equation.*

b. *Verify that the two equations are equivalent.*

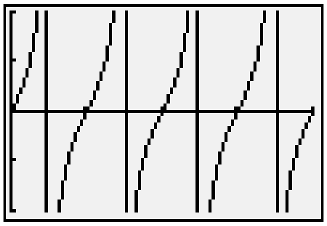
39. $y = \frac{\sin x + \sin 3x}{2\sin 2x}$



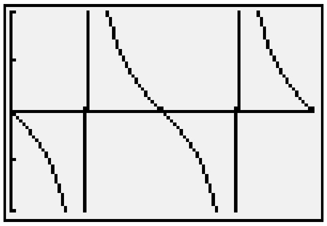
40. $y = \dfrac{\cos x - \cos 3x}{\sin x + \sin 3x}$



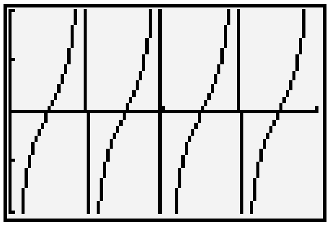
41. $y = \dfrac{\cos x - \cos 5x}{\sin x + \sin 5x}$



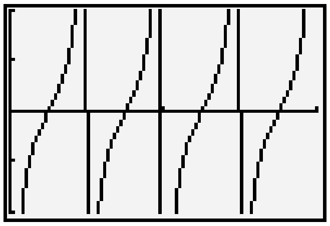
42. $y = \dfrac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x}$



43. $y = \dfrac{\sin x - \sin 3x}{\cos x - \cos 3x}$



44. $y = \dfrac{\sin 2x + \sin 6x}{\cos 6x - \cos 2x}$



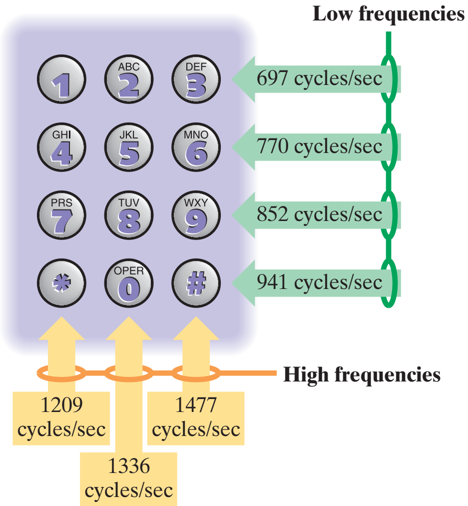
Application Exercises

Use this information to solve Exercises 45–46. The sound produced by touching each button on a touch-tone phone is described by

$$y = \sin 2\pi lt + \sin 2\pi ht,$$

where l and h are the low and high frequencies in the figure shown. For example, what sound is produced by touching 5? The low frequency is $l = 770$ cycles per second and the high frequency is $h = 1336$ cycles per second. The sound produced by touching 5 is described by

$$y = \sin 2\pi(770)t + \sin 2\pi(1336)t.$$



45. The touch-tone phone sequence for that most naive of melodies is given as follows:

Mary Had A Little Lamb

3212333,222,399,3212333322321.

a. Many numbers do not appear in this sequence, including 7. If you accidently touch 7 for one of the notes, describe this sound as the sum of sines.

b. Describe this accidental sound as a product of sines and cosines.

46. The touch-tone phone sequence for *Jingle Bells* is given as follows:

Jingle Bells

333,333,39123,666-663333322329,333,333,39123,666-6633,399621.

a. The first six notes of the song are produced by repeatedly touching 3. Describe this repeated sound as the sum of sines.

b. Describe the repeated sound as a product of sines and cosines.

Writing in Mathematics

In Exercises 47–50, use words to describe the given formula.

47. $\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

48. $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

49. $\sin \alpha + \sin \beta = 2 \sin \dfrac{\alpha + \beta}{2} \cos \dfrac{\alpha - \beta}{2}$

50. $\cos \alpha + \cos \beta = 2 \cos \dfrac{\alpha + \beta}{2} \cos \dfrac{\alpha - \beta}{2}$

51. Describe identities that can be verified using the sum-to-product formulas.

52. Why do the sounds produced by touching each button on a touch-tone phone have the same loudness? Answer the question using the equation described for Exercises 45 and 46, $y = \sin 2\pi lt + \sin 2\pi ht$, and determine the maximum value of y for each sound.

Technology Exercises

In Exercises 53–56, graph each side of the equation in the same viewing rectangle. If the graphs appear to coincide, verify that the equation is an identity. If the graphs do not appear to coincide, find a value of x for which both sides are defined but not equal.

53. $\sin x + \sin 2x = \sin 3x$

54. $\cos x + \cos 2x = \cos 3x$

55. $\sin x + \sin 3x = 2 \sin 2x \cos x$

56. $\cos x + \cos 3x = 2 \cos 2x \cos x$

57. In Exercise 45(a), you wrote an equation for the sound produced by touching 7 on a touch-tone phone. Graph the equation in a [0, 0.01, 0.001] by [−2, 2, 1] viewing rectangle.

58. In Exercise 46(a), you wrote an equation for the sound produced by touching 3 on a touch-tone phone. Graph the equation in a [0, 0.01, 0.001] by [−2, 2, 1] viewing rectangle.

59. In this section, we saw how sums could be expressed as products. Sums of trigonometric functions can also be used to describe functions that are not trigonometric. French mathematician Jean Fourier (1768–1830) showed that *any function* can be described by a series of trigonometric functions. For example, the basic linear function $f(x) = x$ can also be represented by

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right).$$

a. Graph

$$y = 2\left(\frac{\sin x}{1}\right),$$

$$y = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2}\right),$$

$$y = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$$

and

$$y = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{3}\right)$$

in a $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-3, 3, 1]$ viewing rectangle. What patterns do you observe?

b. Graph

$$y = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \frac{\sin 6x}{6} + \frac{\sin 7x}{7} - \frac{\sin 8x}{8} + \frac{\sin 9x}{9} - \frac{\sin 10x}{10}\right)$$

in a $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-3, 3, 1]$ viewing rectangle. Is a portion of the graph beginning to look like the graph of $f(x) = x$? Obtain a better approximation for the line by graphing functions that contain more and more terms involving sines of multiple angles.

c. Use

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

and substitute $\frac{\pi}{2}$ for x to obtain a formula for $\frac{\pi}{2}$. Show at least four nonzero terms. Then multiply both sides of your formula by 2 to write a nonending series of subtractions and additions that approaches π. Use this series to obtain an approximation for π that is more accurate than the one given by your graphing utility.

Critical Thinking Exercises

Make Sense? *In Exercises 60–63, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

60. The product-to-sum formulas are difficult to remember because they are all so similar to one another.

61. I can use the sum and difference formulas for cosines and sines to derive the product-to-sum formulas.

62. I expressed $\sin 13° \cos 48°$ as $\frac{1}{2}(\sin 61° - \sin 35°)$.

63. I expressed $\cos 47° + \cos 59°$ as $2 \cos 53° \cos 6°$.

Use the identities for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$ to solve Exercises 64–65.

64. Add the left and right sides of the identities and derive the product-to-sum formula for $\sin \alpha \cos \beta$.

65. Subtract the left and right sides of the identities and derive the product-to-sum formula for $\cos \alpha \sin \beta$.

In Exercises 66–67, verify the given sum-to-product formula. Start with the right side and obtain the expression on the left side by using an appropriate product-to-sum formula.

66. $\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$

67. $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$

In Exercises 68–69, verify each identity.

68. $\dfrac{\sin 2x + (\sin 3x + \sin x)}{\cos 2x + (\cos 3x + \cos x)} = \tan 2x$

69. $4 \cos x \cos 2x \sin 3x = \sin 2x + \sin 4x + \sin 6x$

Group Exercise

70. This activity should result in an unusual group display entitled "*Frere Jacques*, a New Perspective." Here is the touch-tone phone sequence:

Frere Jacques

4564,4564,69#,69#,#*#964,#*#964,414,414.

Group members should write every sound in the sequence as both the sum of sines and the product of sines and cosines. Use the sum of sines form and a graphing utility with a [0, 0.01, 0.001] by [−2, 2, 1] viewing rectangle to obtain a graph for every sound. Download these graphs. Use the graphs and equations to create your display in such a way that adults find the trigonometry of this naive melody interesting.

Preview Exercises

Exercises 71–73 will help you prepare for the material covered in the next section.

71. Solve: $2(1 - u^2) + 3u = 0$.

72. Solve: $u^3 - 3u = 0$.

73. Solve: $u^2 - u - 1 = 0$.