Group Exercise

67. The group should design five original problems that can be solved using the Laws of Sines and Cosines. At least two problems should be solved using the Law of Sines, one should be the ambiguous case, and at least two problems should be solved using the Law of Cosines. At least one problem should be an application problem using the Law of Sines and at least one problem should involve an application using the Law of Cosines. The group should turn in both the problems and their solutions.

Preview Exercises

Exercises 68–70 will help you prepare for the material covered in the next section.

68. Graph: $y = 3$.

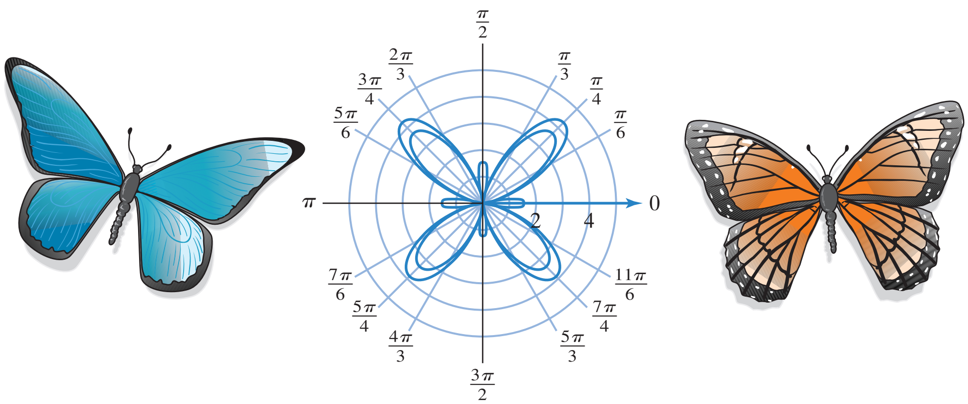
69. Graph: $x^2 + (y - 1)^2 = 1$.

70. Complete the square and write the equation in standard form: $x^2 + 6x + y^2 = 0$. Then give the center and radius of the circle, and graph the equation.

Section 6.3 Polar Coordinates

Objectives

1. Plot points in the polar coordinate system.
2. Find multiple sets of polar coordinates for a given point.
3. Convert a point from polar to rectangular coordinates.
4. Convert a point from rectangular to polar coordinates.
5. Convert an equation from rectangular to polar coordinates.
6. Convert an equation from polar to rectangular coordinates.



Butterflies are among the most celebrated of all insects. It's hard not to notice their beautiful colors and graceful flight. Their symmetry can be explored with trigonometric functions and a system for plotting points called the *polar coordinate system*. In many cases, polar coordinates are simpler and easier to use than rectangular coordinates.

1 Plot points in the polar coordinate system.

Plotting Points in the Polar Coordinate System

The foundation of the polar coordinate system is a horizontal ray that extends to the right. The ray is called the **polar axis** and is shown in **Figure 6.19**. The endpoint of the ray is called the **pole**.

A point P in the polar coordinate system is represented by an ordered pair of numbers (r, θ). **Figure 6.20** shows $P = (r, \theta)$ in the polar coordinate system.

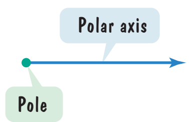


Figure 6.19

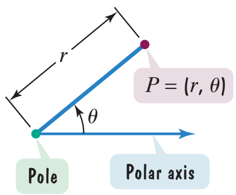


Figure 6.20 Representing a point in the polar coordinate system

- r is a directed distance from the pole to P. (We shall see that r can be positive, negative, or zero.)
- θ is an angle from the polar axis to the line segment from the pole to P. This angle can be measured in degrees or radians. Positive angles are measured counterclockwise from the polar axis. Negative angles are measured clockwise from the polar axis.

We refer to the ordered pair (r, θ) as the **polar coordinates** of P.

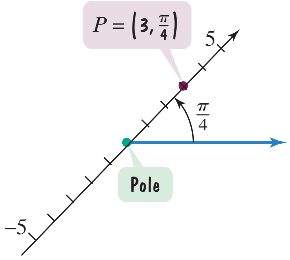


Figure 6.21 Locating a point in polar coordinates

Let's look at a specific example. Suppose that the polar coordinates of a point P are $\left(3, \frac{\pi}{4}\right)$. Because θ is positive, we locate this point by drawing $\theta = \frac{\pi}{4}$ counterclockwise from the polar axis. Then we count out a distance of three units along the terminal side of the angle to reach the point P. **Figure 6.21** shows that $(r, \theta) = \left(3, \frac{\pi}{4}\right)$ lies three units from the pole on the terminal side of the angle $\theta = \frac{\pi}{4}$.

The sign of r is important in locating $P = (r, \theta)$ in polar coordinates.

The Sign of *r* and a Point's Location in Polar Coordinates

The point $P = (r, \theta)$ is located $|r|$ units from the pole. If $r > 0$, the point lies on the terminal side of θ. If $r < 0$, the point lies along the ray opposite the terminal side of θ. If $r = 0$, the point lies at the pole, regardless of the value of θ.

EXAMPLE 1 Plotting Points in a Polar Coordinate System

Plot the points with the following polar coordinates:

a. $(2, 135°)$ **b.** $\left(-3, \frac{3\pi}{2}\right)$ **c.** $\left(-1, -\frac{\pi}{4}\right)$.

Study Tip

Wondering where the concentric circles in **Figure 6.22** came from and why we've shown them? The circles are drawn to help plot each point at the appropriate distance from the pole.

Solution

a. To plot the point $(r, \theta) = (2, 135°)$, begin with the 135° angle. Because 135° is a positive angle, draw $\theta = 135°$ counterclockwise from the polar axis. Now consider $r = 2$. Because $r > 0$, plot the point by going out two units on the terminal side of θ. **Figure 6.22(a)** shows the point.

b. To plot the point $(r, \theta) = \left(-3, \frac{3\pi}{2}\right)$, begin with the $\frac{3\pi}{2}$ angle. Because $\frac{3\pi}{2}$ is a positive angle, we draw $\theta = \frac{3\pi}{2}$ counterclockwise from the polar axis. Now consider $r = -3$. Because $r < 0$, plot the point by going out three units along the ray *opposite* the terminal side of θ. **Figure 6.22(b)** shows the point.

c. To plot the point $(r, \theta) = \left(-1, -\frac{\pi}{4}\right)$, begin with the $-\frac{\pi}{4}$ angle. Because $-\frac{\pi}{4}$ is a negative angle, draw $\theta = -\frac{\pi}{4}$ clockwise from the polar axis. Now consider $r = -1$. Because $r < 0$, plot the point by going out one unit along the ray *opposite* the terminal side of θ. **Figure 6.22(c)** shows the point.

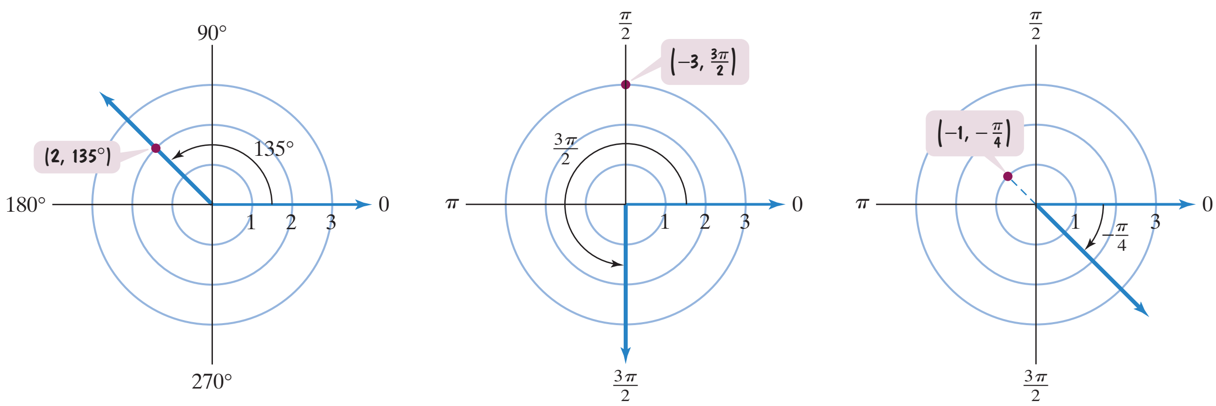


Figure 6.22 Plotting points **(a)** **(b)** **(c)**

Check Point 1 Plot the points with the following polar coordinates:

a. $(3, 315°)$ **b.** $(-2, \pi)$ **c.** $\left(-1, -\frac{\pi}{2}\right)$.

2 Find multiple sets of polar coordinates for a given point.

Multiple Representations of Points in the Polar Coordinate System

In rectangular coordinates, each point (x, y) has exactly one representation. By contrast, any point in polar coordinates can be represented in infinitely many ways. For example,

$$(r, \theta) = (r, \theta + 2\pi) \quad \text{and} \quad (r, \theta) = (-r, \theta + \pi).$$

Adding 1 revolution, or 2π radians, to the angle does not change the point's location.

Adding $\frac{1}{2}$ revolution, or π radians, to the angle and replacing r with $-r$ does not change the point's location.

Thus, to find two other representations for the point (r, θ),

- Add 2π to the angle and do not change r.
- Add π to the angle and replace r with $-r$.

Continually adding or subtracting 2π in either of these representations does not change the point's location.

Discovery

Illustrate the statements in the voice balloons by plotting the points with the following polar coordinates:

a. $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{5\pi}{2}\right)$

b. $\left(3, \frac{\pi}{4}\right)$ and $\left(-3, \frac{5\pi}{4}\right)$.

Multiple Representations of Points

If n is any integer, the point (r, θ) can be represented as

$$(r, \theta) = (r, \theta + 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta + \pi + 2n\pi).$$

EXAMPLE 2 Finding Other Polar Coordinates for a Given Point

The point $\left(2, \frac{\pi}{3}\right)$ is plotted in **Figure 6.23**. Find another representation of this point in which

a. r is positive and $2\pi < \theta < 4\pi$.

b. r is negative and $0 < \theta < 2\pi$.

c. r is positive and $-2\pi < \theta < 0$.

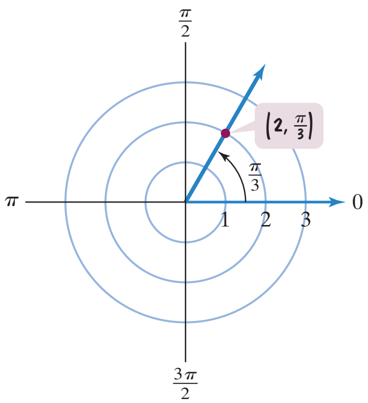


Figure 6.23 Finding other representations of a given point

Solution

a. We want $r > 0$ and $2\pi < \theta < 4\pi$. Using $\left(2, \frac{\pi}{3}\right)$, add 2π to the angle and do not change r.

$$\left(2, \frac{\pi}{3}\right) = \left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{\pi}{3} + \frac{6\pi}{3}\right) = \left(2, \frac{7\pi}{3}\right)$$

b. We want $r < 0$ and $0 < \theta < 2\pi$. Using $\left(2, \frac{\pi}{3}\right)$, add π to the angle and replace r with $-r$.

$$\left(2, \frac{\pi}{3}\right) = \left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{\pi}{3} + \frac{3\pi}{3}\right) = \left(-2, \frac{4\pi}{3}\right)$$

c. We want $r > 0$ and $-2\pi < \theta < 0$. Using $\left(2, \frac{\pi}{3}\right)$, subtract 2π from the angle and do not change r.

$$\left(2, \frac{\pi}{3}\right) = \left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, \frac{\pi}{3} - \frac{6\pi}{3}\right) = \left(2, -\frac{5\pi}{3}\right)$$

Check Point 2 Find another representation of $\left(5, \frac{\pi}{4}\right)$ in which

a. r is positive and $2\pi < \theta < 4\pi$.

b. r is negative and $0 < \theta < 2\pi$.

c. r is positive and $-2\pi < \theta < 0$.

Relations between Polar and Rectangular Coordinates

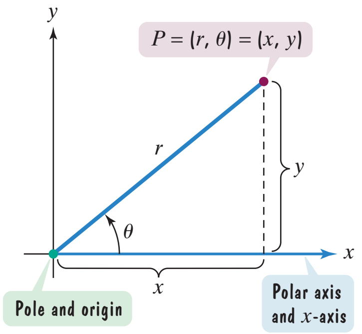


Figure 6.24 Polar and rectangular coordinate systems

We now consider both polar and rectangular coordinates simultaneously. **Figure 6.24** shows the two coordinate systems. The polar axis coincides with the positive x-axis and the pole coincides with the origin. A point P, other than the origin, has rectangular coordinates (x, y) and polar coordinates (r, θ), as indicated in the figure. We wish to find equations relating the two sets of coordinates. From the figure, we see that

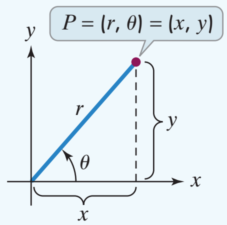
$$x^2 + y^2 = r^2$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}.$$

These relationships hold when P is in any quadrant and when $r > 0$ or $r < 0$.

Relations between Polar and Rectangular Coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$\tan\theta = \frac{y}{x}$$



3 Convert a point from polar to rectangular coordinates.

Point Conversion from Polar to Rectangular Coordinates

To convert a point from polar coordinates (r, θ) to rectangular coordinates (x, y), use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

EXAMPLE 3 Polar-to-Rectangular Point Conversion

Find the rectangular coordinates of the points with the following polar coordinates:

a. $\left(2, \frac{3\pi}{2}\right)$ **b.** $\left(-8, \frac{\pi}{3}\right)$.

Solution We find (x, y) by substituting the given values for r and θ into $x = r\cos\theta$ and $y = r\sin\theta$.

a. We begin with the rectangular coordinates of the point $(r, \theta) = \left(2, \frac{3\pi}{2}\right)$.

$$x = r\cos\theta = 2\cos\frac{3\pi}{2} = 2 \cdot 0 = 0$$

$$y = r\sin\theta = 2\sin\frac{3\pi}{2} = 2(-1) = -2$$

The rectangular coordinates of $\left(2, \frac{3\pi}{2}\right)$ are $(0, -2)$. See **Figure 6.25**.

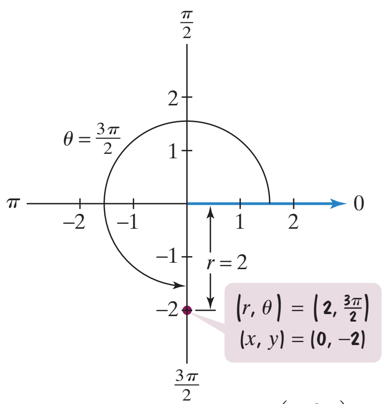


Figure 6.25 Converting $\left(2, \frac{3\pi}{2}\right)$ to rectangular coordinates

b. We now find the rectangular coordinates of the point $(r, \theta) = \left(-8, \frac{\pi}{3}\right)$.

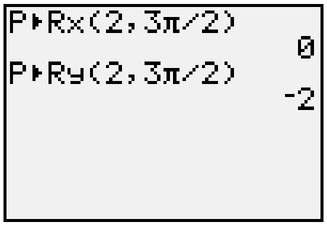
$$x = r\cos\theta = -8\cos\frac{\pi}{3} = -8\left(\frac{1}{2}\right) = -4$$

$$y = r\sin\theta = -8\sin\frac{\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$$

The rectangular coordinates of $\left(-8, \frac{\pi}{3}\right)$ are $\left(-4, -4\sqrt{3}\right)$.

Technology

Some graphing utilities can convert a point from polar coordinates to rectangular coordinates. Consult your manual. The screen on the right verifies the polar-rectangular conversion in Example 3(a). It shows that the rectangular coordinates of $(r, \theta) = \left(2, \frac{3\pi}{2}\right)$ are $(0, -2)$. Notice that the x- and y-coordinates are displayed separately.



Check Point 3 Find the rectangular coordinates of the points with the following polar coordinates:

a. $(3, \pi)$ **b.** $\left(-10, \frac{\pi}{6}\right)$.

4 Convert a point from rectangular to polar coordinates.

Point Conversion from Rectangular to Polar Coordinates

Conversion from rectangular coordinates (x, y) to polar coordinates (r, θ) is a bit more complicated. Keep in mind that there are infinitely many representations for a point in polar coordinates. If the point (x, y) lies in one of the four quadrants, we will use a representation in which

- r is positive, and
- θ is the smallest positive angle with the terminal side passing through (x, y).

These conventions provide the following procedure:

Converting a Point from Rectangular to Polar Coordinates ($r > 0$ and $0 \le \theta < 2\pi$)

1. Plot the point (x, y).
2. Find r by computing the distance from the origin to (x, y): $r = \sqrt{x^2 + y^2}$.
3. Find θ using $\tan\theta = \frac{y}{x}$ with the terminal side of θ passing through (x, y).

EXAMPLE 4 Rectangular-to-Polar Point Conversion

Find polar coordinates of the point whose rectangular coordinates are $\left(-1, \sqrt{3}\right)$.

Solution We begin with $(x, y) = \left(-1, \sqrt{3}\right)$ and use our three-step procedure to find a set of polar coordinates (r, θ).

Step 1 Plot the point (x, y). The point $\left(-1, \sqrt{3}\right)$ is plotted in quadrant II in **Figure 6.26**.

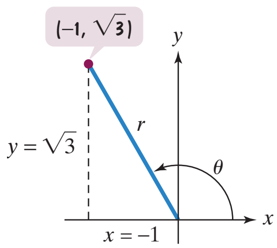


Figure 6.26 Converting $\left(-1, \sqrt{3}\right)$ to polar coordinates

Step 2 Find r by computing the distance from the origin to (x, y).

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$

Step 3 Find θ using $\tan\theta = \frac{y}{x}$ with the terminal side of θ passing through (x, y).

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

We know that $\tan\frac{\pi}{3} = \sqrt{3}$. Because θ lies in quadrant II,

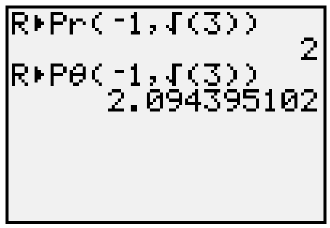
$$\theta = \pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}.$$

One representation of $\left(-1, \sqrt{3}\right)$ in polar coordinates is $(r, \theta) = \left(2, \frac{2\pi}{3}\right)$.

Technology

The screen shows the rectangular-polar conversion for $\left(-1, \sqrt{3}\right)$ on a graphing utility. In Example 4, we showed that $(x, y) = \left(-1, \sqrt{3}\right)$ can be represented in polar coordinates as $(r, \theta) = \left(2, \frac{2\pi}{3}\right)$.

Using $\frac{2\pi}{3} \approx 2.09439510239$ verifies that our conversion is correct. Notice that the r- and (approximate) θ-coordinates are displayed separately.



Check Point 4 Find polar coordinates of the point whose rectangular coordinates are $\left(1, -\sqrt{3}\right)$.

If a point (x, y) lies on a positive or negative axis, we use a representation in which

- r is positive, and
- θ is the smallest quadrantal angle that lies on the same positive or negative axis as (x, y).

In these cases, you can find r and θ by plotting (x, y) and inspecting the figure. Let's see how this is done.

EXAMPLE 5 Rectangular-to-Polar Point Conversion

Find polar coordinates of the point whose rectangular coordinates are $(-2, 0)$.

Solution We begin with $(x, y) = (-2, 0)$ and find a set of polar coordinates (r, θ).

Step 1 Plot the point (x, y). The point $(-2, 0)$ is plotted in **Figure 6.27**.

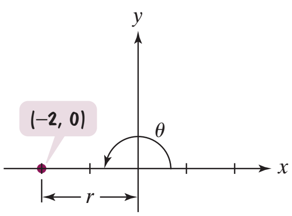


Figure 6.27 Converting $(-2, 0)$ to polar coordinates

Step 2 Find r, the distance from the origin to (x, y). Can you tell by looking at **Figure 6.27** that this distance is 2?

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$$

Step 3 Find θ with θ lying on the same positive or negative axis as (x, y). The point $(-2, 0)$ is on the negative x-axis. Thus, θ lies on the negative x-axis and $\theta = \pi$. One representation of $(-2, 0)$ in polar coordinates is $(2, \pi)$.

Check Point 5 Find polar coordinates of the point whose rectangular coordinates are $(0, -4)$. Express θ in radians.

5 Convert an equation from rectangular to polar coordinates.

Equation Conversion from Rectangular to Polar Coordinates

A **polar equation** is an equation whose variables are r and θ. Two examples of polar equations are

$$r = \frac{5}{\cos\theta + \sin\theta} \quad \text{and} \quad r = 3\csc\theta.$$

To convert a rectangular equation in x and y to a polar equation in r and θ, replace x with $r\cos\theta$ and y with $r\sin\theta$.

EXAMPLE 6 Converting Equations from Rectangular to Polar Coordinates

Convert each rectangular equation to a polar equation that expresses r in terms of θ:

a. $x + y = 5$ **b.** $(x-1)^2 + y^2 = 1$.

Solution Our goal is to obtain equations in which the variables are r and θ, rather than x and y. We use $x = r\cos\theta$ and $y = r\sin\theta$. We then solve the equations for r, obtaining equivalent equations that give r in terms of θ.

a.

$x + y = 5$ — This is the given equation in rectangular coordinates. The graph is a line passing through (5, 0) and (0, 5).

$r\cos\theta + r\sin\theta = 5$ — Replace x with $r\cos\theta$ and y with $r\sin\theta$.

$r(\cos\theta + \sin\theta) = 5$ — Factor out r.

$r = \dfrac{5}{\cos\theta + \sin\theta}$ — Divide both sides of the equation by $\cos\theta + \sin\theta$ and solve for r.

Thus, the polar equation for $x + y = 5$ is $r = \dfrac{5}{\cos\theta + \sin\theta}$.

b.

$(x-1)^2 + y^2 = 1$ — This is the given equation in rectangular coordinates. The graph is a circle with radius 1 and center at $(h, k) = (1, 0)$.

The standard form of a circle's equation is $(x-h)^2 + (y-k)^2 = r^2$, with radius r and center at (h, k).

$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$ — Replace x with $r\cos\theta$ and y with $r\sin\theta$.

$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$ — Use $(A-B)^2 = A^2 - 2AB + B^2$ to square $r\cos\theta - 1$.

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$ — Subtract 1 from both sides and rearrange terms.

$r^2 - 2r\cos\theta = 0$ — Simplify: $r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2 \cdot 1 = r^2$.

$r(r - 2\cos\theta) = 0$ — Factor out r.

$r = 0$ or $r - 2\cos\theta = 0$ — Set each factor equal to 0.

$r = 2\cos\theta$ — Solve for r.

The graph of $r = 0$ is a single point, the pole. Because the pole also satisfies the equation $r = 2\cos\theta$ (for $\theta = \frac{\pi}{2}, r = 0$), it is not necessary to include the equation $r = 0$. Thus, the polar equation for $(x-1)^2 + y^2 = 1$ is $r = 2\cos\theta$.

Check Point 6 Convert each rectangular equation to a polar equation that expresses r in terms of θ:

a. $3x - y = 6$ **b.** $x^2 + (y+1)^2 = 1$.

6 Convert an equation from polar to rectangular coordinates.

Equation Conversion from Polar to Rectangular Coordinates

When we convert an equation from polar to rectangular coordinates, our goal is to obtain an equation in which the variables are x and y, rather than r and θ. We use one or more of the following equations:

$$r^2 = x^2 + y^2 \qquad r\cos\theta = x \qquad r\sin\theta = y \qquad \tan\theta = \frac{y}{x}.$$

To use these equations, it is sometimes necessary to do something to the given polar equation. This could include squaring both sides, using an identity, taking the tangent of both sides, or multiplying both sides by r.

EXAMPLE 7 Converting Equations from Polar to Rectangular Form

Convert each polar equation to a rectangular equation in x and y:

a. $r = 5$ **b.** $\theta = \frac{\pi}{4}$ **c.** $r = 3\csc\theta$ **d.** $r = -6\cos\theta$.

Solution In each case, let's express the rectangular equation in a form that enables us to recognize its graph.

a. We use $r^2 = x^2 + y^2$ to convert the polar equation $r = 5$ to a rectangular equation.

$r = 5$ This is the given polar equation.

$r^2 = 25$ Square both sides.

$x^2 + y^2 = 25$ Use $r^2 = x^2 + y^2$ on the left side.

The rectangular equation for $r = 5$ is $x^2 + y^2 = 25$. The graph is a circle with center at $(0, 0)$ and radius 5.

b. We use $\tan\theta = \frac{y}{x}$ to convert the polar equation $\theta = \frac{\pi}{4}$ to a rectangular equation in x and y.

$\theta = \frac{\pi}{4}$ This is the given polar equation.

$\tan\theta = \tan\frac{\pi}{4}$ Take the tangent of both sides.

$\tan\theta = 1$ $\tan\frac{\pi}{4} = 1$

$\frac{y}{x} = 1$ Use $\tan\theta = \frac{y}{x}$ on the left side.

$y = x$ Multiply both sides by x.

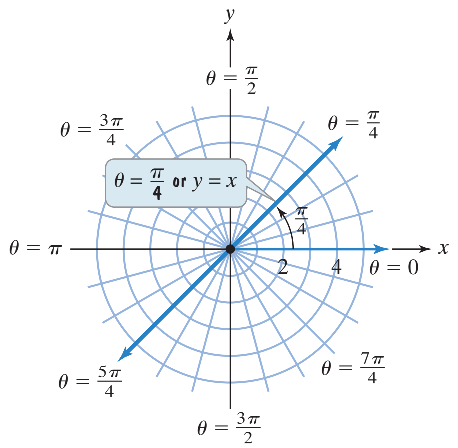


Figure 6.28

The rectangular equation for $\theta = \frac{\pi}{4}$ is $y = x$. The graph is a line that bisects quadrants I and III. **Figure 6.28** shows the line drawn in a polar coordinate system.

c. We use $r\sin\theta = y$ to convert the polar equation $r = 3\csc\theta$ to a rectangular equation. To do this, we express the cosecant in terms of the sine.

$r = 3\csc\theta$ This is the given polar equation.

$r = \frac{3}{\sin\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

$r\sin\theta = 3$ Multiply both sides by $\sin\theta$.

$y = 3$ Use $r\sin\theta = y$ on the left side.

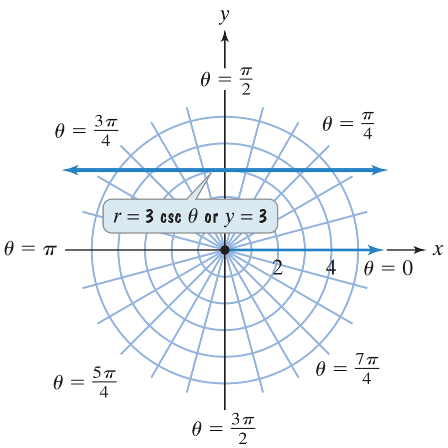


Figure 6.29

The rectangular equation for $r = 3 \csc \theta$ is $y = 3$. The graph is a horizontal line 3 units above the x-axis. **Figure 6.29** shows the line drawn in a polar coordinate system.

d. To convert $r = -6 \cos \theta$ to rectangular coordinates, we multiply both sides by r. Then we use $r^2 = x^2 + y^2$ on the left side and $r \cos \theta = x$ on the right side.

$r = -6 \cos \theta$	This is the given polar equation.
$r^2 = -6r \cos \theta$	Multiply both sides by r.
$x^2 + y^2 = -6x$	Convert to rectangular coordinates: $r^2 = x^2 + y^2$ and $r \cos \theta = x$.
$x^2 + 6x + y^2 = 0$	Add 6x to both sides.
$x^2 + 6x + 9 + y^2 = 9$	Complete the square on x: $\frac{1}{2} \cdot 6 = 3$ and $3^2 = 9$.
$(x + 3)^2 + y^2 = 9$	Factor.

The rectangular equation for $r = -6 \cos \theta$ is $(x + 3)^2 + y^2 = 9$. This last equation is the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, with radius r and center at (h, k). Thus, the graph of $(x + 3)^2 + y^2 = 9$ is a circle with center at $(-3, 0)$ and radius 3.

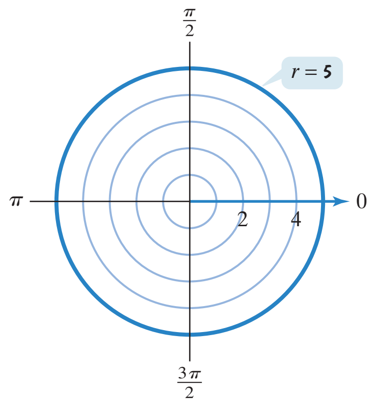


Figure 6.30 The equations $r = 5$ and $x^2 + y^2 = 25$ have the same graph.

Converting a polar equation to a rectangular equation may be a useful way to develop or check a graph. For example, the graph of the polar equation $r = 5$ consists of all points that are five units from the pole. Thus, the graph is a circle centered at the pole with radius 5. The rectangular equation for $r = 5$, namely $x^2 + y^2 = 25$, has precisely the same graph (see **Figure 6.30**). We will discuss graphs of polar equations in the next section.

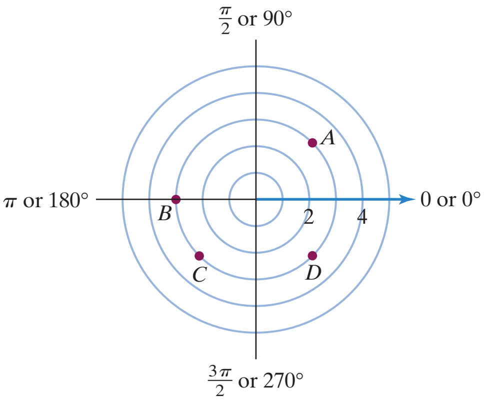
Check Point 7 Convert each polar equation to a rectangular equation in x and y:

a. $r = 4$ **b.** $\theta = \frac{3\pi}{4}$ **c.** $r = -2 \sec \theta$ **d.** $r = 10 \sin \theta$.

Exercise Set 6.3

Practice Exercises

In Exercises 1–10, indicate if the point with the given polar coordinates is represented by A, B, C, or D on the graph.



1. $(3, 225°)$ **2.** $(3, 315°)$ **3.** $\left(-3, \frac{5\pi}{4}\right)$

4. $\left(-3, \frac{\pi}{4}\right)$ **5.** $(3, \pi)$ **6.** $(-3, 0)$

7. $(3, -135°)$ **8.** $(3, -315°)$ **9.** $\left(-3, -\frac{3\pi}{4}\right)$

10. $\left(-3, -\frac{5\pi}{4}\right)$

In Exercises 11–20, use a polar coordinate system like the one shown for Exercises 1–10 to plot each point with the given polar coordinates.

11. $(2, 45°)$ **12.** $(1, 45°)$ **13.** $(3, 90°)$

14. $(2, 270°)$ **15.** $\left(3, \frac{4\pi}{3}\right)$ **16.** $\left(3, \frac{7\pi}{6}\right)$

17. $(-1, \pi)$ **18.** $\left(-1, \frac{3\pi}{2}\right)$ **19.** $\left(-2, -\frac{\pi}{2}\right)$

20. $(-3, -\pi)$

In Exercises 21–26, use a polar coordinate system like the one shown for Exercises 1–10 to plot each point with the given polar coordinates. Then find another representation (r, θ) of this point in which

a. $r > 0, \quad 2\pi < \theta < 4\pi.$

b. $r < 0, \quad 0 < \theta < 2\pi.$

c. $r > 0, \quad -2\pi < \theta < 0.$

21. $\left(5, \frac{\pi}{6}\right)$ **22.** $\left(8, \frac{\pi}{6}\right)$ **23.** $\left(10, \frac{3\pi}{4}\right)$

24. $\left(12, \frac{2\pi}{3}\right)$ **25.** $\left(4, \frac{\pi}{2}\right)$ **26.** $\left(6, \frac{\pi}{2}\right)$

In Exercises 27–32, select the representations that do not change the location of the given point.

27. $(7, 140°)$

a. $(-7, 320°)$ **b.** $(-7, -40°)$

c. $(-7, 220°)$ **d.** $(7, -220°)$

28. $(4, 120°)$

a. $(-4, 300°)$ **b.** $(-4, -240°)$

c. $(4, -240°)$ **d.** $(4, 480°)$

29. $\left(2, -\frac{3\pi}{4}\right)$

a. $\left(2, -\frac{7\pi}{4}\right)$ **b.** $\left(2, \frac{5\pi}{4}\right)$

c. $\left(-2, -\frac{\pi}{4}\right)$ **d.** $\left(-2, -\frac{7\pi}{4}\right)$

30. $\left(-2, \frac{7\pi}{6}\right)$

a. $\left(-2, -\frac{5\pi}{6}\right)$ **b.** $\left(-2, -\frac{\pi}{6}\right)$

c. $\left(2, -\frac{\pi}{6}\right)$ **d.** $\left(2, \frac{\pi}{6}\right)$

31. $\left(-5, -\frac{\pi}{4}\right)$

a. $\left(-5, \frac{7\pi}{4}\right)$ **b.** $\left(5, -\frac{5\pi}{4}\right)$

c. $\left(-5, \frac{11\pi}{4}\right)$ **d.** $\left(5, \frac{\pi}{4}\right)$

32. $(-6, 3\pi)$

a. $(6, 2\pi)$ **b.** $(6, -\pi)$

c. $(-6, \pi)$ **d.** $(-6, -2\pi)$

In Exercises 33–40, polar coordinates of a point are given. Find the rectangular coordinates of each point.

33. $(4, 90°)$ **34.** $(6, 180°)$ **35.** $\left(2, \frac{\pi}{3}\right)$

36. $\left(2, \frac{\pi}{6}\right)$ **37.** $\left(-4, \frac{\pi}{2}\right)$ **38.** $\left(-6, \frac{3\pi}{2}\right)$

39. $(7.4, 2.5)$ **40.** $(8.3, 4.6)$

In Exercises 41–48, the rectangular coordinates of a point are given. Find polar coordinates of each point. Express θ in radians.

41. $(-2, 2)$ **42.** $(2, -2)$

43. $(2, -2\sqrt{3})$ **44.** $(-2\sqrt{3}, 2)$

45. $(-\sqrt{3}, -1)$ **46.** $(-1, -\sqrt{3})$

47. $(5, 0)$ **48.** $(0, -6)$

In Exercises 49–58, convert each rectangular equation to a polar equation that expresses r in terms of θ.

49. $3x + y = 7$ **50.** $x + 5y = 8$

51. $x = 7$ **52.** $y = 3$

53. $x^2 + y^2 = 9$ **54.** $x^2 + y^2 = 16$

55. $(x - 2)^2 + y^2 = 4$ **56.** $x^2 + (y + 3)^2 = 9$

57. $y^2 = 6x$ **58.** $x^2 = 6y$

In Exercises 59–74, convert each polar equation to a rectangular equation. Then use a rectangular coordinate system to graph the rectangular equation.

59. $r = 8$ **60.** $r = 10$

61. $\theta = \frac{\pi}{2}$ **62.** $\theta = \frac{\pi}{3}$

63. $r \sin \theta = 3$ **64.** $r \cos \theta = 7$

65. $r = 4 \csc \theta$ **66.** $r = 6 \sec \theta$

67. $r = \sin \theta$ **68.** $r = \cos \theta$

69. $r = 12 \cos \theta$ **70.** $r = -4 \sin \theta$

71. $r = 6 \cos \theta + 4 \sin \theta$ **72.** $r = 8 \cos \theta + 2 \sin \theta$

73. $r^2 \sin 2\theta = 2$ **74.** $r^2 \sin 2\theta = 4$

Practice Plus

In Exercises 75–78, show that each statement is true by converting the given polar equation to a rectangular equation.

75. Show that the graph of $r = a \sec \theta$ is a vertical line a units to the right of the y-axis if $a > 0$ and $|a|$ units to the left of the y-axis if $a < 0$.

76. Show that the graph of $r = a \csc \theta$ is a horizontal line a units above the x-axis if $a > 0$ and $|a|$ units below the x-axis if $a < 0$.

77. Show that the graph of $r = a \sin \theta$ is a circle with center at $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$.

78. Show that the graph of $r = a \cos \theta$ is a circle with center at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

In Exercises 79–80, convert each polar equation to a rectangular equation. Then determine the graph's slope and y-intercept.

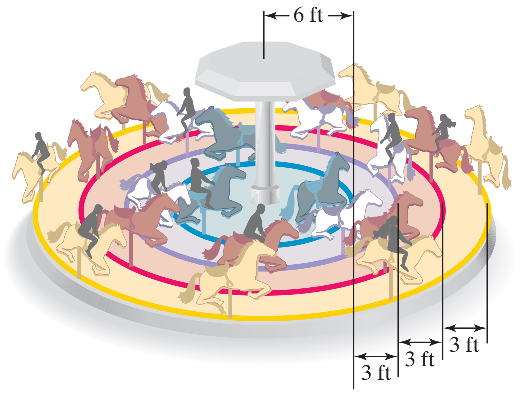
79. $r \sin\left(\theta - \frac{\pi}{4}\right) = 2$ **80.** $r \cos\left(\theta + \frac{\pi}{6}\right) = 8$

In Exercises 81–82, find the rectangular coordinates of each pair of points. Then find the distance, in simplified radical form, between the points.

81. $\left(2, \frac{2\pi}{3}\right)$ and $\left(4, \frac{\pi}{6}\right)$ **82.** $(6, \pi)$ and $\left(5, \frac{7\pi}{4}\right)$

Application Exercises

Use the figure of the merry-go-round to solve Exercises 83–84. There are four circles of horses. Each circle is three feet from the next circle. The radius of the inner circle is 6 feet.



83. If a horse in the outer circle is $\frac{2}{3}$ of the way around the merry-go-round, give its polar coordinates.

84. If a horse in the inner circle is $\frac{5}{6}$ of the way around the merry-go-round, give its polar coordinates.

The wind is blowing at 10 knots. Sailboat racers look for a sailing angle to the 10-knot wind that produces maximum sailing speed. In this application, (r, θ) describes the sailing speed, r, in knots, at an angle θ to the 10-knot wind. Use this information to solve Exercises 85–87.

85. Interpret the polar coordinates: $(6.3, 50°)$.

86. Interpret the polar coordinates: $(7.4, 85°)$.

87. Four points in this 10-knot-wind situation are $(6.3, 50°)$, $(7.4, 85°)$, $(7.5, 105°)$, and $(7.3, 135°)$. Based on these points, which sailing angle to the 10-knot wind would you recommend to a serious sailboat racer? What sailing speed is achieved at this angle?

Writing in Mathematics

88. Explain how to plot (r, θ) if $r > 0$ and $\theta > 0$.

89. Explain how to plot (r, θ) if $r < 0$ and $\theta > 0$.

90. If you are given polar coordinates of a point, explain how to find two additional sets of polar coordinates for the point.

91. Explain how to convert a point from polar to rectangular coordinates. Provide an example with your explanation.

92. Explain how to convert a point from rectangular to polar coordinates. Provide an example with your explanation.

93. Explain how to convert from a rectangular equation to a polar equation.

94. In converting $r = 5$ from a polar equation to a rectangular equation, describe what should be done to both sides of the equation and why this should be done.

95. In converting $r = \sin\theta$ from a polar equation to a rectangular equation, describe what should be done to both sides of the equation and why this should be done.

96. Suppose that (r, θ) describes the sailing speed, r, in knots, at an angle θ to a wind blowing at 20 knots. You have a list of all ordered pairs (r, θ) for integral angles from $\theta = 0°$ to $\theta = 180°$. Describe a way to present this information so that a serious sailboat racer can visualize sailing speeds at different sailing angles to the wind.

Technology Exercises

In Exercises 97–99, polar coordinates of a point are given. Use a graphing utility to find the rectangular coordinates of each point to three decimal places.

97. $\left(4, \frac{2\pi}{3}\right)$

98. $(5.2, 1.7)$

99. $(-4, 1.088)$

In Exercises 100–102, the rectangular coordinates of a point are given. Use a graphing utility in radian mode to find polar coordinates of each point to three decimal places.

100. $(-5, 2)$

101. $(\sqrt{5}, 2)$

102. $(-4.308, -7.529)$

Critical Thinking Exercises

Make Sense? *In Exercises 103–106, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

103. I must have made a mistake because my polar representation of a given point is not the same as the answer in the back of the book.

104. When converting a point from polar coordinates to rectangular coordinates, there are infinitely many possible rectangular coordinate pairs.

105. After plotting the point with rectangular coordinates $(0, -4)$, I found polar coordinates without having to show any work.

106. When I convert an equation from polar form to rectangular form, the rectangular equation might not define y as a function of x.

107. Prove that the distance, d, between two points with polar coordinates (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

108. Use the formula in Exercise 107 to find the distance between $\left(2, \frac{5\pi}{6}\right)$ and $\left(4, \frac{\pi}{6}\right)$. Express the answer in simplified radical form.

Preview Exercises

Exercises 109–111 will help you prepare for the material covered in the next section. In each exercise, use a calculator to complete the table of coordinates. Where necessary, round to two decimal places. Then plot the resulting points, (r, θ), using a polar coordinate system.

109.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$r = 1 - \cos\theta$							

110.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$r = 1 + 2\sin\theta$										

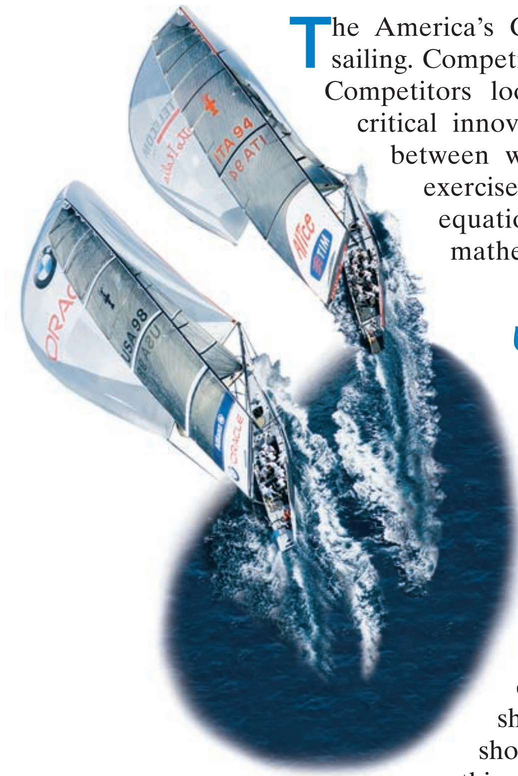
111.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 4\sin 2\theta$									

Section 6.4 Graphs of Polar Equations

Objectives

1. Use point plotting to graph polar equations.
2. Use symmetry to graph polar equations.



The America's Cup is the supreme event in ocean sailing. Competition is fierce and the costs are huge. Competitors look to mathematics to provide the critical innovation that can make the difference between winning and losing. In this section's exercise set, you will see how graphs of polar equations play a role in sailing faster using mathematics.

Using Polar Grids to Graph Polar Equations

Recall that a **polar equation** is an equation whose variables are r and θ. The **graph of a polar equation** is the set of all points whose polar coordinates satisfy the equation. We use **polar grids** like the one shown in **Figure 6.31** to graph polar equations. The grid consists of circles with centers at the pole. This polar grid shows five such circles. A polar grid also shows lines passing through the pole. In this grid, each line represents an angle for which we know the exact values of the trigonometric functions.

Many polar coordinate grids show more circles and more lines through the pole than in **Figure 6.31**. See if your campus bookstore has paper with polar grids and use the polar graph paper throughout this section.

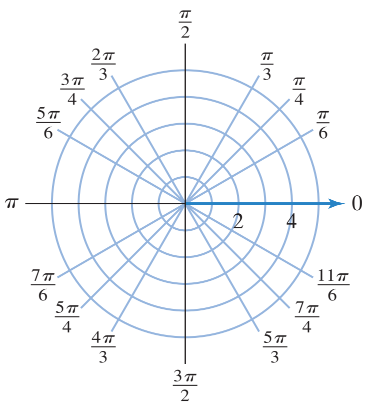


Figure 6.31 A polar coordinate grid

1 Use point plotting to graph polar equations.

Graphing a Polar Equation by Point Plotting

One method for graphing a polar equation such as $r = 4\cos\theta$ is the **point-plotting method**. First, we make a table of values that satisfy the equation. Next, we plot these ordered pairs as points in the polar coordinate system. Finally, we connect the points with a smooth curve. This often gives us a picture of all ordered pairs (r, θ) that satisfy the equation.

EXAMPLE 1 **Graphing an Equation Using the Point-Plotting Method**

Graph the polar equation $r = 4\cos\theta$ with θ in radians.