In Exercises 18–20, convert each rectangular equation to a polar equation that expresses r in terms of θ.

18. $5x - y = 7$ **19.** $y = -7$

20. $(x + 1)^2 + y^2 = 1$

In Exercises 21–25, convert each polar equation to a rectangular equation. Then use your knowledge of the rectangular equation to graph the polar equation in a polar coordinate system.

21. $r = 6$ **22.** $\theta = \frac{\pi}{3}$

23. $r = -3 \csc \theta$ **24.** $r = -10 \cos \theta$

25. $r = 4 \sin \theta \sec^2 \theta$

In Exercises 26–27, test for symmetry with respect to

a. *the polar axis.* **b.** *the line* $\theta = \frac{\pi}{2}$. **c.** *the pole.*

26. $r = 1 - 4 \cos \theta$ **27.** $r^2 = 4 \cos 2\theta$

In Exercises 28–32, graph each polar equation. Be sure to test for symmetry.

28. $r = -4 \sin \theta$ **29.** $r = 2 - 2 \cos \theta$

30. $r = 2 - 4 \cos \theta$ **31.** $r = 2 \sin 3\theta$

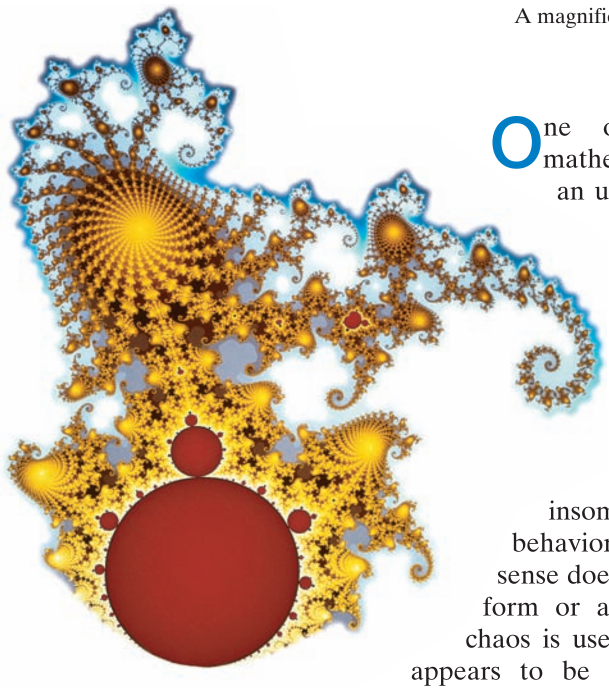
32. $r^2 = 16 \sin 2\theta$

Section 6.5

Complex Numbers in Polar Form; DeMoivre's Theorem

Objectives

1. Plot complex numbers in the complex plane.
2. Find the absolute value of a complex number.
3. Write complex numbers in polar form.
4. Convert a complex number from polar to rectangular form.
5. Find products of complex numbers in polar form.
6. Find quotients of complex numbers in polar form.
7. Find powers of complex numbers in polar form.
8. Find roots of complex numbers in polar form.



A magnification of the Mandelbrot set

One of the new frontiers of mathematics suggests that there is an underlying order in things that appear to be random, such as the hiss and crackle of background noises as you tune a radio. Irregularities in the heartbeat, some of them severe enough to cause a heart attack, or irregularities in our sleeping patterns, such as insomnia, are examples of chaotic behavior. Chaos in the mathematical sense does not mean a complete lack of form or arrangement. In mathematics, chaos is used to describe something that appears to be random but is not actually random. The patterns of chaos appear in images like the one shown above, called the Mandelbrot set. Magnified portions of this image yield repetitions of the original structure, as well as new and unexpected patterns. The Mandelbrot set transforms the hidden structure of chaotic events into a source of wonder and inspiration.

The Mandelbrot set is made possible by opening up graphing to include complex numbers in the form $a + bi$, where $i = \sqrt{-1}$. In this section, you will learn how to graph complex numbers and write them in terms of trigonometric functions.

1 Plot complex numbers in the complex plane.

The Complex Plane

We know that a real number can be represented as a point on a number line. By contrast, a complex number $z = a + bi$ is represented as a point (a, b) in a coordinate plane, as shown in **Figure 6.38** at the top of the next page. The horizontal axis of the coordinate plane is called the **real axis**. The vertical axis is called the **imaginary axis**. The coordinate system is called the **complex plane**. Every complex number corresponds to a point in the complex plane and every point in the complex plane corresponds to a complex number. When we represent a complex number as a point in the complex plane, we say that we are **plotting the complex number**.

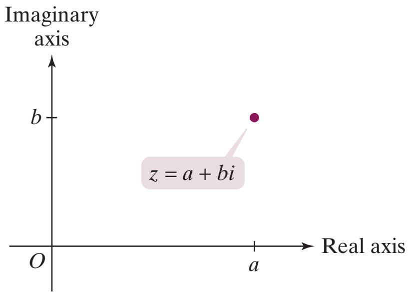


Figure 6.38 Plotting $z = a + bi$ in the complex plane

EXAMPLE 1 Plotting Complex Numbers

Plot each complex number in the complex plane:

a. $z = 3 + 4i$ **b.** $z = -1 - 2i$ **c.** $z = -3$ **d.** $z = -4i$.

Solution See **Figure 6.39**.

a. We plot the complex number $z = 3 + 4i$ the same way we plot $(3, 4)$ in the rectangular coordinate system. We move three units to the right on the real axis and four units up parallel to the imaginary axis.

b. The complex number $z = -1 - 2i$ corresponds to the point $(-1, -2)$ in the rectangular coordinate system. Plot the complex number by moving one unit to the left on the real axis and two units down parallel to the imaginary axis.

c. Because $z = -3 = -3 + 0i$, this complex number corresponds to the point $(-3, 0)$. We plot -3 by moving three units to the left on the real axis.

d. Because $z = -4i = 0 - 4i$, this number corresponds to the point $(0, -4)$. We plot the complex number by moving four units down on the imaginary axis.

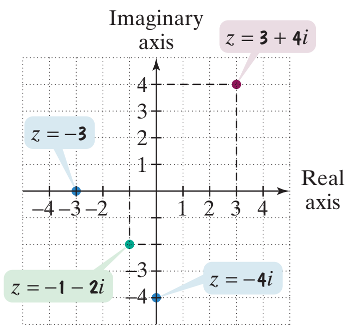


Figure 6.39 Plotting complex numbers

Check Point 1 Plot each complex number in the complex plane:

a. $z = 2 + 3i$ **b.** $z = -3 - 5i$ **c.** $z = -4$ **d.** $z = -i$.

2 Find the absolute value of a complex number.

Recall that the absolute value of a real number is its distance from 0 on a number line. The **absolute value of the complex number** $z = a + bi$, denoted by $|z|$, is the distance from the origin to the point z in the complex plane. **Figure 6.40** illustrates that we can use the Pythagorean Theorem to represent $|z|$ in terms of a and b: $|z| = \sqrt{a^2 + b^2}$.

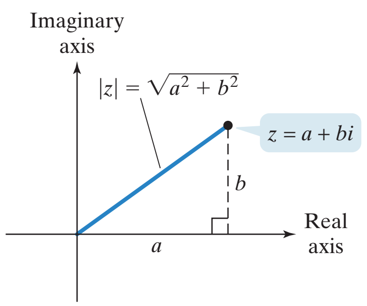


Figure 6.40

The Absolute Value of a Complex Number

The **absolute value** of the complex number $a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

EXAMPLE 2 Finding the Absolute Value of a Complex Number

Determine the absolute value of each of the following complex numbers:

a. $z = 3 + 4i$ **b.** $z = -1 - 2i$.

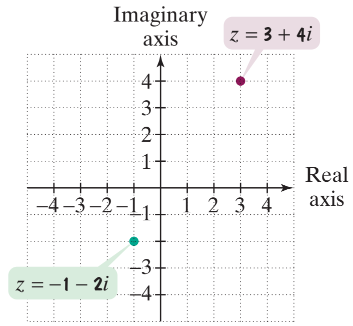


Figure 6.41

Solution

a. The absolute value of $z = 3 + 4i$ is found using $a = 3$ and $b = 4$.

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$ Use $z = \sqrt{a^2 + b^2}$ with $a = 3$ and $b = 4$.

Thus, the distance from the origin to the point $z = 3 + 4i$, shown in quadrant I in **Figure 6.41**, is five units.

b. The absolute value of $z = -1 - 2i$ is found using $a = -1$ and $b = -2$.

$$|z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$$ Use $z = \sqrt{a^2 + b^2}$ with $a = -1$ and $b = -2$.

Thus, the distance from the origin to the point $z = -1 - 2i$, shown in quadrant III in **Figure 6.41**, is $\sqrt{5}$ units.

Check Point 2 Determine the absolute value of each of the following complex numbers:

a. $z = 5 + 12i$ **b.** $2 - 3i$.

3 Write complex numbers in polar form.

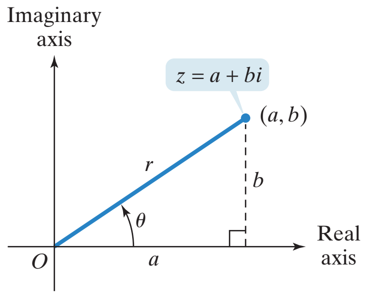


Figure 6.42

Polar Form of a Complex Number

A complex number in the form $z = a + bi$ is said to be in **rectangular form**. Suppose that its absolute value is r. In **Figure 6.42**, we let θ be an angle in standard position whose terminal side passes through the point (a, b). From the figure, we see that

$$r = \sqrt{a^2 + b^2}.$$

Likewise, according to the definitions of the trigonometric functions,

$$\cos\theta = \frac{a}{r} \qquad \sin\theta = \frac{b}{r} \qquad \tan\theta = \frac{b}{a}.$$

$$a = r\cos\theta \qquad b = r\sin\theta$$

By substituting the expressions for a and b into $z = a + bi$, we write the complex number in terms of trigonometric functions.

$$z = a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

$a = r\cos\theta$ and $b = r\sin\theta$.

Factor out r from each of the two previous terms.

The expression $z = r(\cos\theta + i\sin\theta)$ is called the **polar form of a complex number**.

Polar Form of a Complex Number

The complex number $z = a + bi$ is written in **polar form** as

$$z = r(\cos\theta + i\sin\theta),$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$. The value of r is called the **modulus** (plural: moduli) of the complex number z and the angle θ is called the **argument** of the complex number z with $0 \le \theta < 2\pi$.

EXAMPLE 3 Writing a Complex Number in Polar Form

Plot $z = -2 - 2i$ in the complex plane. Then write z in polar form.

Solution The complex number $z = -2 - 2i$ is in rectangular form $z = a + bi$, with $a = -2$ and $b = -2$. We plot the number by moving two units to the left on the real axis and two units down parallel to the imaginary axis, as shown in **Figure 6.43** on the next page.

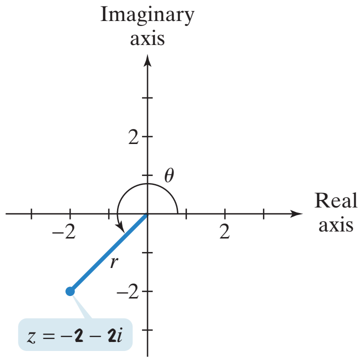


Figure 6.43 Plotting $z = -2 - 2i$ and writing the number in polar form

By definition, the polar form of z is $r(\cos\theta + i\sin\theta)$. We need to determine the value for r, the modulus, and the value for θ, the argument. **Figure 6.43** shows r and θ. We use $r = \sqrt{a^2 + b^2}$ with $a = -2$ and $b = -2$ to find r.

$$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = \sqrt{4\cdot 2} = 2\sqrt{2}$$

We use $\tan\theta = \dfrac{b}{a}$ with $a = -2$ and $b = -2$ to find θ.

$$\tan\theta = \frac{b}{a} = \frac{-2}{-2} = 1$$

We know that $\tan\dfrac{\pi}{4} = 1$. **Figure 6.43** shows that the argument, θ, satisfying $\tan\theta = 1$ lies in quadrant III. Thus,

$$\theta = \pi + \frac{\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4} = \frac{5\pi}{4}.$$

We use $r = 2\sqrt{2}$ and $\theta = \dfrac{5\pi}{4}$ to write the polar form. The polar form of $z = -2 - 2i$ is

$$z = r(\cos\theta + i\sin\theta) = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$

Check Point 3 Plot $z = -1 - i\sqrt{3}$ in the complex plane. Then write z in polar form. Express the argument in radians. (We write $-1 - i\sqrt{3}$, rather than $-1 - \sqrt{3}\,i$, which could easily be confused with $-1 - \sqrt{3i}$.)

4 Convert a complex number from polar to rectangular form.

EXAMPLE 4 Writing a Complex Number in Rectangular Form

Write $z = 2(\cos 60° + i\sin 60°)$ in rectangular form.

Solution The complex number $z = 2(\cos 60° + i\sin 60°)$ is in polar form, with $r = 2$ and $\theta = 60°$. We use exact values for $\cos 60°$ and $\sin 60°$ to write the number in rectangular form.

$$2(\cos 60° + i\sin 60°) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

The rectangular form of $z = 2(\cos 60° + i\sin 60°)$ is

$$z = 1 + i\sqrt{3}.$$

Write i before the radical to avoid confusion with $\sqrt{3i}$.

Check Point 4 Write $z = 4(\cos 30° + i\sin 30°)$ in rectangular form.

5 Find products of complex numbers in polar form.

Products and Quotients in Polar Form

We can multiply and divide complex numbers fairly quickly if the numbers are expressed in polar form.

Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their product, $z_1 z_2$, is

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

To multiply two complex numbers, multiply moduli and add arguments.

To prove this result, we begin by multiplying using the FOIL method. Then we simplify the product using the sum formulas for sine and cosine.

$$\begin{aligned}
z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\
&= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) && \text{Rearrange factors.} \\
&= r_1 r_2(\underbrace{\cos\theta_1\cos\theta_2}_{\text{F}} + \underbrace{i\cos\theta_1\sin\theta_2}_{\text{O}} + \underbrace{i\sin\theta_1\cos\theta_2}_{\text{I}} + \underbrace{i^2\sin\theta_1\sin\theta_2}_{\text{L}}) && \text{Use the FOIL method.} \\
&= r_1 r_2[\cos\theta_1\cos\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) + i^2\sin\theta_1\sin\theta_2] && \text{Factor } i \text{ from the second and third terms.} \\
&= r_1 r_2[\cos\theta_1\cos\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) - \sin\theta_1\sin\theta_2] && i^2 = -1 \\
&= r_1 r_2[\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\text{This is } \cos(\theta_1+\theta_2).} + i\underbrace{(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)}_{\text{This is } \sin(\theta_1+\theta_2).}] && \text{Rearrange terms.} \\
&= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]
\end{aligned}$$

This result gives a rule for finding the product of two complex numbers in polar form. The two parts to the rule are shown in the following voice balloons.

$$z_1 z_2 = \underbrace{r_1 r_2}_{\text{Multiply moduli.}}[\cos\underbrace{(\theta_1 + \theta_2)}_{\text{Add arguments.}} + i\sin\underbrace{(\theta_1 + \theta_2)}_{\text{Add arguments.}}]$$

EXAMPLE 5 Finding Products of Complex Numbers in Polar Form

Find the product of the complex numbers. Leave the answer in polar form.

$$z_1 = 4(\cos 50° + i\sin 50°) \qquad z_2 = 7(\cos 100° + i\sin 100°)$$

Solution

$$\begin{aligned}
&z_1 z_2 \\
&= [4(\cos 50° + i\sin 50°)][7(\cos 100° + i\sin 100°)] && \text{Form the product of the given numbers.} \\
&= (4\cdot 7)[\cos(50° + 100°) + i\sin(50° + 100°)] && \text{Multiply moduli and add arguments.} \\
&= 28(\cos 150° + i\sin 150°) && \text{Simplify.}
\end{aligned}$$

Check Point 5 Find the product of the complex numbers. Leave the answer in polar form.

$$z_1 = 6(\cos 40° + i\sin 40°) \qquad z_2 = 5(\cos 20° + i\sin 20°)$$

6 Find quotients of complex numbers in polar form.

Using algebraic methods for dividing complex numbers and the difference formulas for sine and cosine, we can obtain a rule for dividing complex numbers in polar form. The proof of this rule can be found in Appendix A, or you can derive the rule on your own by working Exercise 110 in this section's exercise set.

Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their quotient, $\frac{z_1}{z_2}$, is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

To divide two complex numbers, divide moduli and subtract arguments.

EXAMPLE 6 Finding Quotients of Complex Numbers in Polar Form

Find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave the answer in polar form.

$$z_1 = 12\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \qquad z_2 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution

$$\frac{z_1}{z_2} = \frac{12\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} \qquad \text{Form the quotient of the given numbers.}$$

$$= \frac{12}{4}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\right] \qquad \text{Divide moduli and subtract arguments.}$$

$$= 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \qquad \text{Simplify: } \frac{3\pi}{4} - \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Check Point 6 Find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave the answer in polar form.

$$z_1 = 50\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

7 Find powers of complex numbers in polar form.

Powers of Complex Numbers in Polar Form

We can use a formula to find powers of complex numbers if the complex numbers are expressed in polar form. This formula can be illustrated by repeatedly multiplying by $r(\cos\theta + i\sin\theta)$.

$$z = r(\cos\theta + i\sin\theta) \qquad \text{Start with } z.$$

$$z \cdot z = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) \qquad \text{Multiply } z \text{ by } z = r(\cos\theta + i\sin\theta).$$

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta) \qquad \text{Multiply moduli: } r \cdot r = r^2. \text{ Add arguments: } \theta + \theta = 2\theta.$$

$$z^2 \cdot z = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) \qquad \text{Multiply } z^2 \text{ by } z = r(\cos\theta + i\sin\theta).$$

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta) \qquad \text{Multiply moduli: } r^2 \cdot r = r^3. \text{ Add arguments: } 2\theta + \theta = 3\theta.$$

$$z^3 \cdot z = r^3(\cos 3\theta + i\sin 3\theta)r(\cos\theta + i\sin\theta) \qquad \text{Multiply } z^3 \text{ by } z = r(\cos\theta + i\sin\theta).$$

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta) \qquad \text{Multiply moduli: } r^3 \cdot r = r^4. \text{ Add arguments: } 3\theta + \theta = 4\theta.$$

Do you see a pattern forming? If n is a positive integer, it appears that z^n is obtained by raising the modulus to the nth power and multiplying the argument by n. The formula for the nth power of a complex number is known as **DeMoivre's Theorem** in honor of the French mathematician Abraham DeMoivre (1667–1754).

DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If n is a positive integer, then z to the nth power, z^n, is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

EXAMPLE 7 Finding the Power of a Complex Number

Find $[2(\cos 20° + i\sin 20°)]^6$. Write the answer in rectangular form, $a + bi$.

Solution We begin by applying DeMoivre's Theorem.

$[2(\cos 20° + i\sin 20°)]^6$

$= 2^6[\cos(6 \cdot 20°) + i\sin(6 \cdot 20°)]$ — Raise the modulus to the 6th power and multiply the argument by 6.

$= 64(\cos 120° + i\sin 120°)$ — Simplify.

$= 64\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ — Write the answer in rectangular form.

$= -32 + 32i\sqrt{3}$ — Multiply and express the answer in $a + bi$ form.

Check Point 7 Find $[2(\cos 30° + i\sin 30°)]^5$. Write the answer in rectangular form.

EXAMPLE 8 Finding the Power of a Complex Number

Find $(1 + i)^8$ using DeMoivre's Theorem. Write the answer in rectangular form, $a + bi$.

Solution DeMoivre's Theorem applies to complex numbers in polar form. Thus, we must first write $1 + i$ in $r(\cos\theta + i\sin\theta)$ form. Then we can use DeMoivre's Theorem. The complex number $1 + i$ is plotted in **Figure 6.44**. From the figure, we obtain values for r and θ.

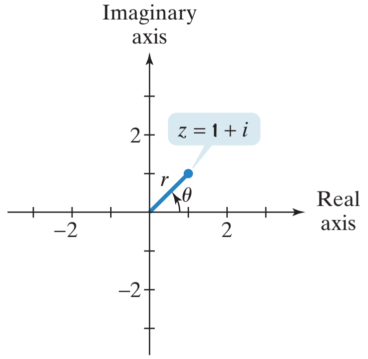


Figure 6.44 Plotting $1 + i$ and writing the number in polar form

$$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1 \text{ and } \theta = \frac{\pi}{4} \text{ because } \theta \text{ lies in quadrant I.}$$

Using these values,

$$1 + i = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Now we use DeMoivre's Theorem to raise $1 + i$ to the 8th power.

$(1 + i)^8$

$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^8$ — Work with the polar form of $1 + i$.

$= (\sqrt{2})^8\left[\cos\left(8 \cdot \frac{\pi}{4}\right) + i\sin\left(8 \cdot \frac{\pi}{4}\right)\right]$ — Apply DeMoivre's Theorem. Raise the modulus to the 8th power and multiply the argument by 8.

$= 16(\cos 2\pi + i\sin 2\pi)$ — Simplify: $(\sqrt{2})^8 = (2^{1/2})^8 = 2^4 = 16$.

$= 16(1 + 0i)$ — $\cos 2\pi = 1$ and $\sin 2\pi = 0$.

$= 16$ or $16 + 0i$ — Simplify.

Check Point 8 Find $(1 + i)^4$ using DeMoivre's Theorem. Write the answer in rectangular form.

8 Find roots of complex numbers in polar form.

Roots of Complex Numbers in Polar Form

In Example 7, we showed that

$$[2(\cos 20° + i \sin 20°)]^6 = 64(\cos 120° + i \sin 120°).$$

We say that $2(\cos 20° + i \sin 20°)$ is a **complex sixth root** of $64(\cos 120° + i \sin 120°)$. It is one of six distinct complex sixth roots of $64(\cos 120° + i \sin 120°)$.

In general, if a complex number z satisfies the equation

$$z^n = w,$$

we say that z is a **complex *n*th root** of w. It is one of n distinct nth complex roots that can be found using the following theorem:

DeMoivre's Theorem for Finding Complex Roots

Let $w = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, w has n distinct complex nth roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{(radians)}$$

$$\text{or} \quad z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right] \text{(degrees)},$$

where $k = 0, 1, 2, \ldots, n - 1$.

By raising the radian or degree formula for z_k to the nth power, you can use DeMoivre's Theorem for powers to show that $z_k^n = w$. Thus, each z_k is a complex nth root of w.

DeMoivre's Theorem for finding complex roots states that every complex number has two distinct complex square roots, three distinct complex cube roots, four distinct complex fourth roots, and so on. Each root has the same modulus, $\sqrt[n]{r}$. Successive roots have arguments that differ by the same amount, $\frac{2\pi}{n}$ or $\frac{360°}{n}$. This means that if you plot all the complex roots of any number, they will be equally spaced on a circle centered at the origin, with radius $\sqrt[n]{r}$.

EXAMPLE 9 Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 120° + i \sin 120°)$. Write roots in polar form, with θ in degrees.

Solution There are exactly four fourth roots of the given complex number. From DeMoivre's Theorem for finding complex roots, the fourth roots of $16(\cos 120° + i \sin 120°)$ are

$$z_k = \sqrt[4]{16}\left[\cos\left(\frac{120° + 360°k}{4}\right) + i\sin\left(\frac{120° + 360°k}{4}\right)\right], \quad k = 0, 1, 2, 3.$$

Use $z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$.
In $16(\cos 120° + i \sin 120°)$, $r = 16$ and $\theta = 120°$.
Because we are finding fourth roots, $n = 4$.

The four fourth roots are found by substituting 0, 1, 2, and 3 for k in the expression for z_k, repeated in the margin. Thus, the four complex fourth roots are as follows:

$$z_k = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + 360^\circ k}{4}\right) + i\sin\left(\frac{120^\circ + 360^\circ k}{4}\right)\right]$$

The formula for the four fourth roots of $16(\cos 120^\circ + i \sin 120^\circ)$, $k = 0, 1, 2, 3$ (repeated)

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + 360^\circ \cdot 0}{4}\right) + i\sin\left(\frac{120^\circ + 360^\circ \cdot 0}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{120^\circ}{4} + i\sin\frac{120^\circ}{4}\right) = 2(\cos 30^\circ + i \sin 30^\circ)$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + 360^\circ \cdot 1}{4}\right) + i\sin\left(\frac{120^\circ + 360^\circ \cdot 1}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{480^\circ}{4} + i\sin\frac{480^\circ}{4}\right) = 2(\cos 120^\circ + i \sin 120^\circ)$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + 360^\circ \cdot 2}{4}\right) + i\sin\left(\frac{120^\circ + 360^\circ \cdot 2}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{840^\circ}{4} + i\sin\frac{840^\circ}{4}\right) = 2(\cos 210^\circ + i \sin 210^\circ)$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + 360^\circ \cdot 3}{4}\right) + i\sin\left(\frac{120^\circ + 360^\circ \cdot 3}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{1200^\circ}{4} + i\sin\frac{1200^\circ}{4}\right) = 2(\cos 300^\circ + i \sin 300^\circ).$$

In **Figure 6.45**, we have plotted each of the four fourth roots of $16(\cos 120^\circ + i \sin 120^\circ)$. Notice that they are equally spaced at 90° intervals on a circle with radius 2.

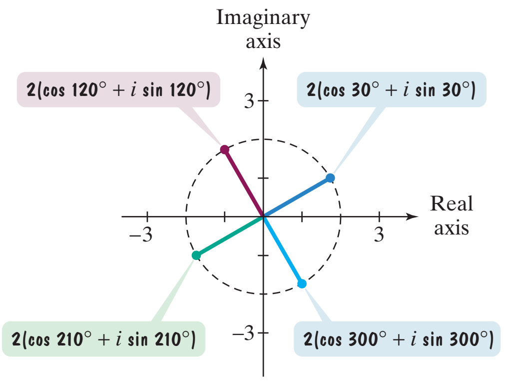


Figure 6.45 Plotting the four fourth roots of $16(\cos 120^\circ + i \sin 120^\circ)$

Check Point 9 Find all the complex fourth roots of $16(\cos 60^\circ + i \sin 60^\circ)$. Write roots in polar form, with θ in degrees.

EXAMPLE 10 Finding the Roots of a Complex Number

Find all the cube roots of 8. Write roots in rectangular form.

Solution DeMoivre's Theorem for roots applies to complex numbers in polar form. Thus, we will first write 8, or $8 + 0i$, in polar form. We express θ in radians, although degrees could also be used.

$$8 = r(\cos\theta + i\sin\theta) = 8(\cos 0 + i \sin 0)$$

There are exactly three cube roots of 8. From DeMoivre's Theorem for finding complex roots, the cube roots of 8 are

$$z_k = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi k}{3}\right) + i\sin\left(\frac{0+2\pi k}{3}\right)\right],\quad k = 0, 1, 2.$$

Use $z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$.
In $8(\cos 0 + i\sin 0)$, $r = 8$ and $\theta = 0$.
Because we are finding cube roots, $n = 3$.

The three cube roots of 8 are found by substituting 0, 1, and 2 for k in the expression for z_k above the voice balloon. Thus, the three cube roots of 8 are

$$z_0 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi\cdot 0}{3}\right) + i\sin\left(\frac{0+2\pi\cdot 0}{3}\right)\right]$$

$$= 2(\cos 0 + i\sin 0) = 2(1 + i\cdot 0) = 2$$

$$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi\cdot 1}{3}\right) + i\sin\left(\frac{0+2\pi\cdot 1}{3}\right)\right]$$

$$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$$

$$z_2 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi\cdot 2}{3}\right) + i\sin\left(\frac{0+2\pi\cdot 2}{3}\right)\right]$$

$$= 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} + i\cdot\left(-\frac{\sqrt{3}}{2}\right)\right) = -1 - i\sqrt{3}.$$

The three cube roots of 8 are plotted in **Figure 6.46.**

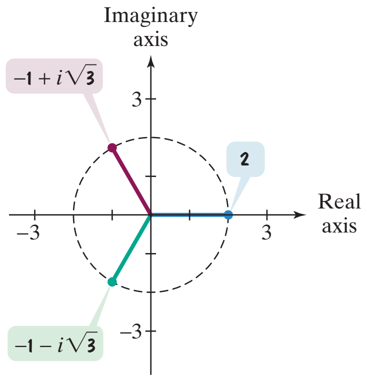


Figure 6.46 The three cube roots of 8 are equally spaced at intervals of $\frac{2\pi}{3}$ on a circle with radius 2.

Discovery

Use DeMoivre's Theorem to cube $-1 + i\sqrt{3}$ or $-1 - i\sqrt{3}$ and obtain 8.

Check Point 10 Find all the cube roots of 27. Write roots in rectangular form.

The Mandelbrot Set

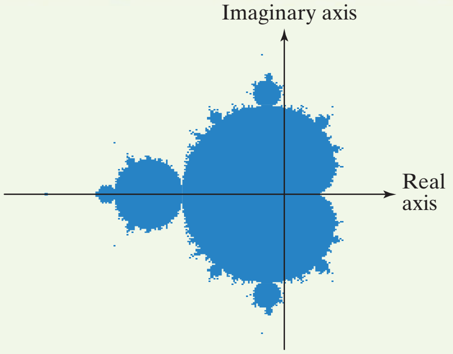


Figure 6.47

The set of all complex numbers for which the sequence

$$z, z^2 + z, (z^2 + z)^2 + z, [(z^2 + z)^2 + z]^2 + z, \ldots$$

is bounded is called the **Mandelbrot set**. Plotting these complex numbers in the complex plane results in a graph that is "buglike" in shape, shown in **Figure 6.47**. Colors can be added to the boundary of the graph. At the boundary, color choices depend on how quickly the numbers in the boundary approach infinity when substituted into the sequence shown. The magnified boundary is shown in the introduction to this section. It includes the original buglike structure, as well as new and interesting patterns. With each level of magnification, repetition and unpredictable formations interact to create what has been called the most complicated mathematical object ever known.

Exercise Set 6.5

Practice Exercises

In Exercises 1–10, plot each complex number and find its absolute value.

1. $z = 4i$
2. $z = 3i$
3. $z = 3$
4. $z = 4$
5. $z = 3 + 2i$
6. $z = 2 + 5i$
7. $z = 3 - i$
8. $z = 4 - i$
9. $z = -3 + 4i$
10. $z = -3 - 4i$

In Exercises 11–26, plot each complex number. Then write the complex number in polar form. You may express the argument in degrees or radians.

11. $2 + 2i$
12. $1 + i\sqrt{3}$
13. $-1 - i$
14. $2 - 2i$
15. $-4i$
16. $-3i$
17. $2\sqrt{3} - 2i$
18. $-2 + 2i\sqrt{3}$
19. -3
20. -4
21. $-3\sqrt{2} - 3i\sqrt{3}$
22. $3\sqrt{2} - 3i\sqrt{2}$
23. $-3 + 4i$
24. $-2 + 3i$
25. $2 - i\sqrt{3}$
26. $1 - i\sqrt{5}$

In Exercises 27–36, write each complex number in rectangular form. If necessary, round to the nearest tenth.

27. $6(\cos 30° + i \sin 30°)$
28. $12(\cos 60° + i \sin 60°)$
29. $4(\cos 240° + i \sin 240°)$
30. $10(\cos 210° + i \sin 210°)$
31. $8\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$
32. $4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
33. $5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$
34. $7\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$
35. $20(\cos 205° + i \sin 205°)$
36. $30(\cos 2.3 + i \sin 2.3)$

In Exercises 37–44, find the product of the complex numbers. Leave answers in polar form.

37. $z_1 = 6(\cos 20° + i \sin 20°)$
    $z_2 = 5(\cos 50° + i \sin 50°)$
38. $z_1 = 4(\cos 15° + i \sin 15°)$
    $z_2 = 7(\cos 25° + i \sin 25°)$
39. $z_1 = 3\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)$
    $z_2 = 4\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)$
40. $z_1 = 3\left(\cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8}\right)$
    $z_2 = 10\left(\cos \frac{\pi}{16} + i \sin \frac{\pi}{16}\right)$
41. $z_1 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$
    $z_2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$
42. $z_1 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$
    $z_2 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$
43. $z_1 = 1 + i$
    $z_2 = -1 + i$
44. $z_1 = 1 + i$
    $z_2 = 2 + 2i$

In Exercises 45–52, find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave answers in polar form. In Exercises 49–50, express the argument as an angle between 0° and 360°.

45. $z_1 = 20(\cos 75° + i \sin 75°)$
    $z_2 = 4(\cos 25° + i \sin 25°)$
46. $z_1 = 50(\cos 80° + i \sin 80°)$
    $z_2 = 10(\cos 20° + i \sin 20°)$
47. $z_1 = 3\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)$
    $z_2 = 4\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)$
48. $z_1 = 3\left(\cos \frac{5\pi}{18} + i \sin \frac{5\pi}{18}\right)$
    $z_2 = 10\left(\cos \frac{\pi}{16} + i \sin \frac{\pi}{16}\right)$
49. $z_1 = \cos 80° + i \sin 80°$
    $z_2 = \cos 200° + i \sin 200°$
50. $z_1 = \cos 70° + i \sin 70°$
    $z_2 = \cos 230° + i \sin 230°$
51. $z_1 = 2 + 2i$
    $z_2 = 1 + i$
52. $z_1 = 2 - 2i$
    $z_2 = 1 - i$

In Exercises 53–64, use DeMoivre's Theorem to find the indicated power of the complex number. Write answers in rectangular form.

53. $[4(\cos 15° + i \sin 15°)]^3$
54. $[2(\cos 10° + i \sin 10°)]^3$
55. $[2(\cos 80° + i \sin 80°)]^3$
56. $[2(\cos 40° + i \sin 40°)]^3$
57. $\left[\frac{1}{2}\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)\right]^6$
58. $\left[\frac{1}{2}\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)\right]^5$
59. $\left[\sqrt{2}\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)\right]^4$
60. $\left[\sqrt{3}\left(\cos \frac{5\pi}{18} + i \sin \frac{5\pi}{18}\right)\right]^6$
61. $(1 + i)^5$
62. $(1 - i)^5$
63. $(\sqrt{3} - i)^6$
64. $(\sqrt{2} - i)^4$

In Exercises 65–68, find all the complex roots. Write roots in polar form with θ in degrees.

65. The complex square roots of $9(\cos 30° + i \sin 30°)$
66. The complex square roots of $25(\cos 210° + i \sin 210°)$
67. The complex cube roots of $8(\cos 210° + i \sin 210°)$
68. The complex cube roots of $27(\cos 306° + i \sin 306°)$

In Exercises 69–76, find all the complex roots. Write roots in rectangular form. If necessary, round to the nearest tenth.

69. The complex fourth roots of $81\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

70. The complex fifth roots of $32\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

71. The complex fifth roots of 32

72. The complex sixth roots of 64

73. The complex cube roots of 1

74. The complex cube roots of i

75. The complex fourth roots of $1 + i$

76. The complex fifth roots of $-1 + i$

Practice Plus

In Exercises 77–80, convert to polar form and then perform the indicated operations. Express answers in polar and rectangular form.

77. $i(2 + 2i)\left(-\sqrt{3} + i\right)$

78. $(1 + i)\left(1 - i\sqrt{3}\right)\left(-\sqrt{3} + i\right)$

79. $\dfrac{\left(1 + i\sqrt{3}\right)(1 - i)}{2\sqrt{3} - 2i}$ **80.** $\dfrac{\left(-1 + i\sqrt{3}\right)\left(2 - 2i\sqrt{3}\right)}{4\sqrt{3} - 4i}$

In Exercises 81–86, solve each equation in the complex number system. Express solutions in polar and rectangular form.

81. $x^6 - 1 = 0$ **82.** $x^6 + 1 = 0$

83. $x^4 + 16i = 0$ **84.** $x^5 - 32i = 0$

85. $x^3 - \left(1 + i\sqrt{3}\right) = 0$ **86.** $x^3 - \left(1 - i\sqrt{3}\right) = 0$

In calculus, it can be shown that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

In Exercises 87–90, use this result to plot each complex number.

87. $e^{\frac{\pi i}{4}}$ **88.** $e^{\frac{\pi i}{6}}$ **89.** $-e^{-\pi i}$ **90.** $-2e^{-2\pi i}$

Application Exercises

In Exercises 91–92, show that the given complex number z plots as a point in the Mandelbrot set.

a. *Write the first six terms of the sequence*

$$z_1, z_2, z_3, z_4, z_5, z_6, \ldots$$

where

$z_1 = z$: *Write the given number.*

$z_2 = z^2 + z$: *Square z_1 and add the given number.*

$z_3 = (z^2 + z)^2 + z$: *Square z_2 and add the given number.*

$z_4 = [(z^2 + z)^2 + z]^2 + z$: *Square z_3 and add the given number.*

z_5: *Square z_4 and add the given number.*

z_6: *Square z_5 and add the given number.*

b. *If the sequence that you began writing in part (a) is bounded, the given complex number belongs to the Mandelbrot set. Show that the sequence is bounded by writing two complex numbers. One complex number should be greater in absolute value than the absolute values of the terms in the sequence. The second complex number should be less in absolute value than the absolute values of the terms in the sequence.*

91. $z = i$ **92.** $z = -i$

Writing in Mathematics

93. Explain how to plot a complex number in the complex plane. Provide an example with your explanation.

94. How do you determine the absolute value of a complex number?

95. What is the polar form of a complex number?

96. If you are given a complex number in rectangular form, how do you write it in polar form?

97. If you are given a complex number in polar form, how do you write it in rectangular form?

98. Explain how to find the product of two complex numbers in polar form.

99. Explain how to find the quotient of two complex numbers in polar form.

100. Explain how to find the power of a complex number in polar form.

101. Explain how to use DeMoivre's Theorem for finding complex roots to find the two square roots of 9.

102. Describe the graph of all complex numbers with an absolute value of 6.

103. The image of the Mandelbrot set in the section opener exhibits self-similarity: Magnified portions repeat much of the pattern of the whole structure, as well as new and unexpected patterns. Describe an object in nature that exhibits self-similarity.

Technology Exercises

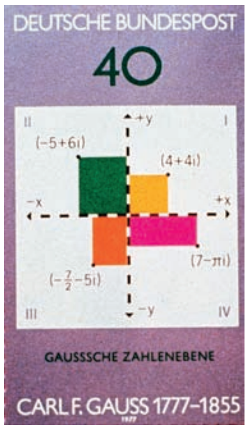
104. Use the rectangular-to-polar feature on a graphing utility to verify any four of your answers in Exercises 11–26. Be aware that you may have to adjust the angle for the correct quadrant.

105. Use the polar-to-rectangular feature on a graphing utility to verify any four of your answers in Exercises 27–36.

Critical Thinking Exercises

Make Sense? *In Exercises 106–109, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

106. This stamp, honoring the work done by the German mathematician Carl Friedrich Gauss (1777–1855) with complex numbers, illustrates that a complex number $a + bi$ can be interpreted geometrically as the point (a, b) in the xy-plane.



107. I multiplied two complex numbers in polar form by first multiplying the moduli and then multiplying the arguments.

108. The proof of the formula for the product of two complex numbers in polar form uses the sum formulas for cosines and sines that I studied in the previous chapter.

109. My work with complex numbers verified that the only possible cube root of 8 is 2.

110. Prove the rule for finding the quotient of two complex numbers in polar form. Begin the proof as follows, using the conjugate of the denominator's second factor:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}.$$

Perform the indicated multiplications. Then use the difference formulas for sine and cosine.

111. Plot each of the complex fourth roots of 1.

Group Exercise

112. Group members should prepare and present a seminar on mathematical chaos. Include one or more of the following topics in your presentation: fractal images, the role of complex numbers in generating fractal images, algorithms, iterations, iteration number, and fractals in nature. Be sure to include visual images that will intrigue your audience.

Preview Exercises

Exercises 113–115 will help you prepare for the material covered in the next section.

113. Use the distance formula to determine if the line segment with endpoints $(-3, -3)$ and $(0, 3)$ has the same length as the line segment with endpoints $(0, 0)$ and $(3, 6)$.

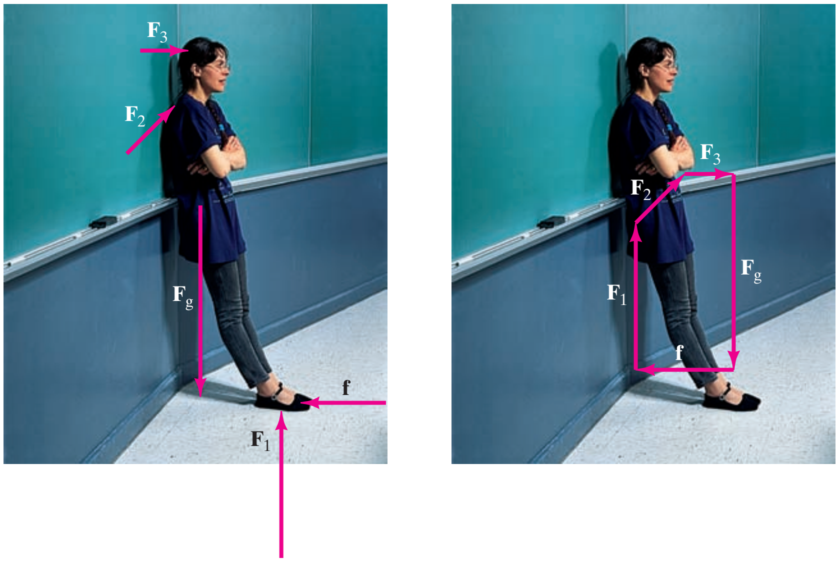
114. Use slope to determine if the line through $(-3, -3)$ and $(0, 3)$ is parallel to the line through $(0, 0)$ and $(3, 6)$.

115. Simplify: $4(5x + 4y) - 2(6x - 9y)$.

Section 6.6 Vectors

Objectives

1. Use magnitude and direction to show vectors are equal.
2. Visualize scalar multiplication, vector addition, and vector subtraction as geometric vectors.
3. Represent vectors in the rectangular coordinate system.
4. Perform operations with vectors in terms of **i** and **j**.
5. Find the unit vector in the direction of **v**.
6. Write a vector in terms of its magnitude and direction.
7. Solve applied problems involving vectors.



It's been a dynamic lecture, but now that it's over it's obvious that my professor is exhausted. She's slouching motionless against the board and—what's that? The forces acting against her body, including the pull of gravity, are appearing as arrows. I know that mathematics reveals the hidden patterns of the universe, but this is ridiculous. Does the arrangement of the arrows on the right have anything to do with the fact that my wiped-out professor is not sliding down the wall?

Ours is a world of pushes and pulls. For example, suppose you are pulling a cart up a 30° incline, requiring an effort of 100 pounds. This quantity is described by giving its magnitude (a number indicating size, including a unit of measure) and also its direction. The magnitude is 100 pounds and the direction is 30° from the horizontal. Quantities that involve both a magnitude and a direction are called **vector quantities**, or **vectors** for short. Here is another example of a vector:

> You are driving due north at 50 miles per hour. The magnitude is the speed, 50 miles per hour. The direction of motion is due north.