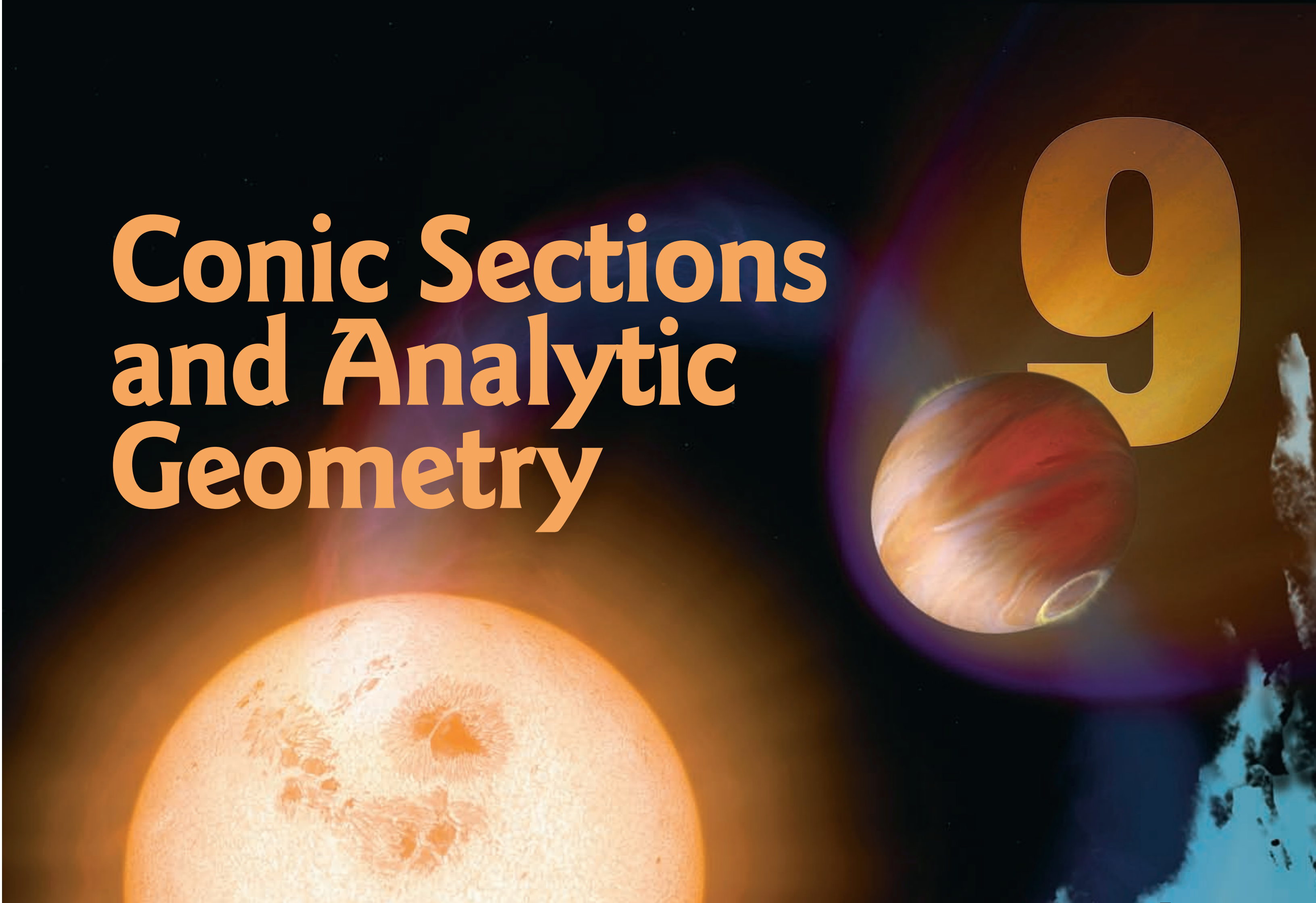


9

Conic Sections and Analytic Geometry

From ripples in water to the path on which humanity journeys through space, certain curves occur naturally throughout the universe. Over two thousand years ago, the ancient Greeks studied these curves, called *conic sections*, without regard to their immediate usefulness simply because studying them elicited ideas that were exciting, challenging, and interesting. The ancient Greeks could not have imagined the applications of these curves in the twenty-first century. They enable the Hubble Space Telescope, a large satellite about the size of a school bus orbiting 375 miles above Earth, to gather distant rays of light and focus them into spectacular images of our evolving universe. They provide doctors with a procedure for dissolving kidney stones painlessly without invasive surgery. In this chapter, we use the rectangular coordinate system to study the conic sections and the mathematics behind their surprising applications.

Here's where you'll find applications that move beyond planet Earth:

- Planetary orbits: Section 9.1, page 881; Exercise Set 9.1, Exercise 78.
- Halley's Comet: Essay on page 882.
- Hubble Space Telescope: Section 9.3, pages 900 and 908.

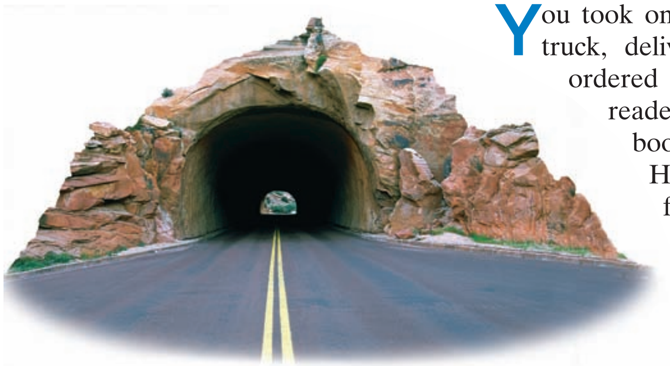
For a kidney stone here on Earth, see Section 9.1, page 882.

Section 9.1

The Ellipse

Objectives

1. Graph ellipses centered at the origin.
2. Write equations of ellipses in standard form.
3. Graph ellipses not centered at the origin.
4. Solve applied problems involving ellipses.



You took on a summer job driving a truck, delivering books that were ordered online. You're an avid reader, so just being around books sounded appealing. However, now you're feeling a bit shaky driving the truck for the first time. It's 10 feet wide and 9 feet high; compared to your compact car, it feels like you're behind the wheel of a tank. Up ahead you see a sign at the semielliptical entrance to a tunnel: Caution! Tunnel is 10 Feet High at Center Peak. Then you see another sign: Caution! Tunnel Is 40 Feet Wide. Will your truck clear the opening of the tunnel's archway?

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. **Figure 9.1** illustrates the four conic sections: the circle, the ellipse, the parabola, and the hyperbola.

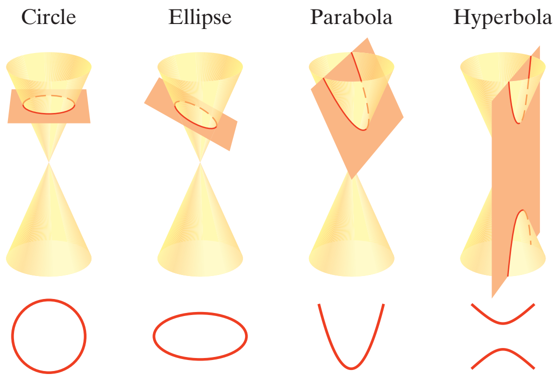


Figure 9.1 Obtaining the conic sections by intersecting a plane and a cone

In this section, we study the symmetric oval-shaped curve known as the ellipse. We will use a geometric definition for an ellipse to derive its equation. With this equation, we will determine if your delivery truck will clear the tunnel's entrance.

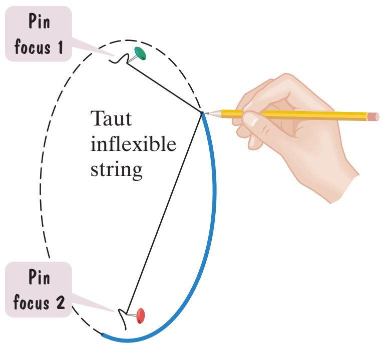


Figure 9.2 Drawing an ellipse

Definition of an Ellipse

Figure 9.2 illustrates how to draw an ellipse. Place pins at two fixed points, each of which is called a *focus* (plural: *foci*). If the ends of a fixed length of string are fastened to the pins and we draw the string taut with a pencil, the path traced by the pencil will be an ellipse. Notice that the sum of the distances of the pencil point from the foci remains constant because the length of the string is fixed. This procedure for drawing an ellipse illustrates its geometric definition.

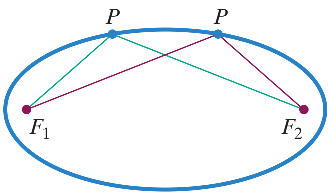


Figure 9.3

Definition of an Ellipse

An **ellipse** is the set of all points, P, in a plane the sum of whose distances from two fixed points, F_1 and F_2, is constant (see **Figure 9.3**). These two fixed points are called the **foci** (plural of **focus**). The midpoint of the segment connecting the foci is the **center** of the ellipse.

Figure 9.4 illustrates that an ellipse can be elongated in any direction. In this section, we will limit our discussion to ellipses that are elongated horizontally or vertically. The line through the foci intersects the ellipse at two points, called the **vertices** (singular: **vertex**). The line segment that joins the vertices is the **major axis**. Notice that the midpoint of the major axis is the center of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis at the center is called the **minor axis** of the ellipse.

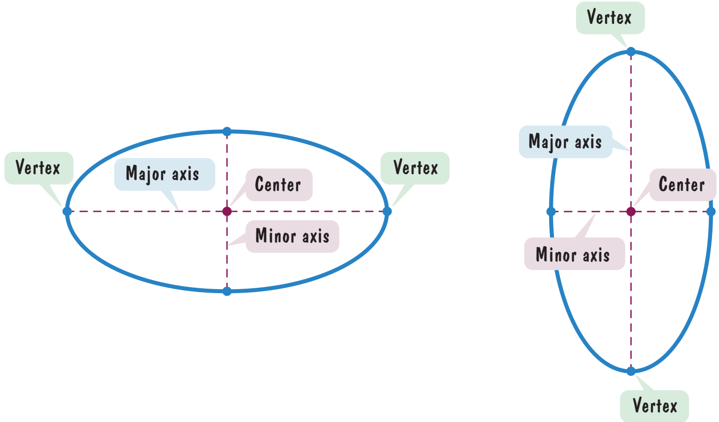


Figure 9.4 Horizontal and vertical elongations of an ellipse

Standard Form of the Equation of an Ellipse

The rectangular coordinate system gives us a unique way of describing an ellipse. It enables us to translate an ellipse's geometric definition into an algebraic equation.

We start with **Figure 9.5** to obtain an ellipse's equation. We've placed an ellipse that is elongated horizontally into a rectangular coordinate system. The foci are on the x-axis at $(-c, 0)$ and $(c, 0)$, as in **Figure 9.5**. In this way, the center of the ellipse is at the origin. We let (x, y) represent the coordinates of any point on the ellipse.

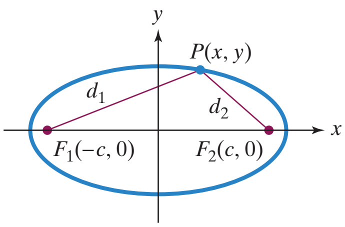


Figure 9.5

Study Tip

The algebraic details behind eliminating the radicals and obtaining the equation shown can be found in Appendix A. There you will find a step-by-step derivation of the ellipse's equation.

What does the definition of an ellipse tell us about the point (x, y) in **Figure 9.5**? For any point (x, y) on the ellipse, the sum of the distances to the two foci, $d_1 + d_2$, must be constant. As we shall see, it is convenient to denote this constant by $2a$. Thus, the point (x, y) is on the ellipse if and only if

$$d_1 + d_2 = 2a.$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \quad \text{Use the distance formula.}$$

After eliminating radicals and simplifying, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Look at the triangle in **Figure 9.5**. Notice that the distance from F_1 to F_2 is $2c$. Because the length of any side of a triangle is less than the sum of the lengths of the other two sides, $2c < d_1 + d_2$. Equivalently, $2c < 2a$ and $c < a$. Consequently, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Substituting b^2 for $a^2 - c^2$ in the preceding equation, we obtain

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{Simplify.}$$

This last equation is the **standard form of the equation of an ellipse centered at the origin**. There are two such equations, one for a horizontal major axis and one for a vertical major axis.

Study Tip

The form $c^2 = a^2 - b^2$ is the one you should remember. When finding the foci, this form is easy to manipulate.

Standard Forms of the Equations of an Ellipse

The **standard form of the equation of an ellipse** with center at the origin, and major and minor axes of lengths $2a$ and $2b$ (where a and b are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Figure 9.6 illustrates that the vertices are on the major axis, a units from the center. The foci are on the major axis, c units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$.

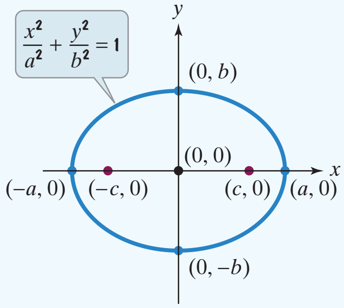


Figure 9.6(a) Major axis is horizontal with length $2a$.

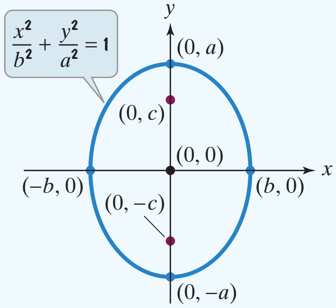


Figure 9.6(b) Major axis is vertical with length $2a$.

The intercepts shown in **Figure 9.6(a)** can be obtained algebraically. Let's do this for

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

***x*-intercepts: Set $y = 0$.**

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$x = \pm a$$

x-intercepts are $-a$ and a. The graph passes through $(-a, 0)$ and $(a, 0)$, which are the vertices.

***y*-intercepts: Set $x = 0$.**

$$\frac{y^2}{b^2} = 1$$

$$y^2 = b^2$$

$$y = \pm b$$

y-intercepts are $-b$ and b. The graph passes through $(0, -b)$ and $(0, b)$.

Using the Standard Form of the Equation of an Ellipse

We can use the standard form of an ellipse's equation to graph the ellipse. Although the definition of the ellipse is given in terms of its foci, the foci are not part of the graph. A complete graph of an ellipse can be obtained without graphing the foci.

1 Graph ellipses centered at the origin.

EXAMPLE 1 Graphing an Ellipse Centered at the Origin

Graph and locate the foci: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

Solution The given equation is the standard form of an ellipse's equation with $a^2 = 9$ and $b^2 = 4$.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$a^2 = 9$. This is the larger of the two denominators.

$b^2 = 4$. This is the smaller of the two denominators.

Technology

We graph $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with a graphing utility by solving for y.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right)$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}}$$

Notice that the square root property requires us to define two functions. Enter

$$y_1 = 2 \boxed{\sqrt{\ }} (1 \boxed{-} x \boxed{\wedge} 2 \boxed{\div} 9)$$

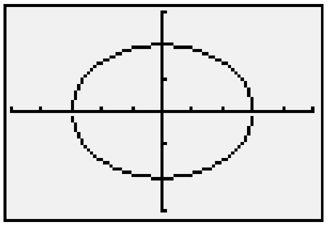
and

$$y_2 = -y_1.$$

To see the true shape of the ellipse, use the

$\boxed{\text{ZOOM SQUARE}}$

feature so that one unit on the y-axis is the same length as one unit on the x-axis.



[−5, 5, 1] by [−3, 3, 1]

Because the denominator of the x^2-term is greater than the denominator of the y^2-term, the major axis is horizontal. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 9$, $a = 3$. Thus, the vertices are $(-3, 0)$ and $(3, 0)$, shown in **Figure 9.7**.

Now let us find the endpoints of the vertical minor axis. According to the standard form of the equation, these endpoints are $(0, -b)$ and $(0, b)$. Because $b^2 = 4$, $b = 2$. Thus, the endpoints of the minor axis are $(0, -2)$ and $(0, 2)$. They are shown in **Figure 9.7**.

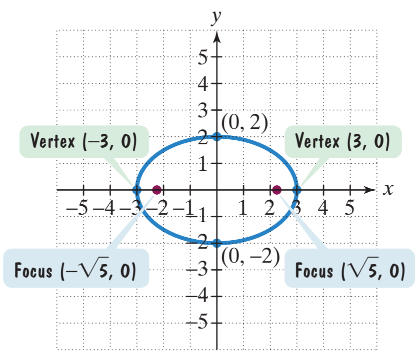


Figure 9.7 The graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Finally, we find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 9$ and $b^2 = 4$. Thus,

$$c^2 = a^2 - b^2 = 9 - 4 = 5.$$

Because $c^2 = 5$, $c = \sqrt{5}$. The foci, $(-c, 0)$ and $(c, 0)$, are located at $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, 0\right)$. They are shown in **Figure 9.7**.

You can sketch the ellipse in **Figure 9.7** by locating endpoints on the major and minor axes.

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Endpoints of the major axis are 3 units to the right and left of the center.

Endpoints of the minor axis are 2 units up and down from the center.

Check Point 1 Graph and locate the foci: $\frac{x^2}{36} + \frac{y^2}{9} = 1.$

EXAMPLE 2 Graphing an Ellipse Centered at the Origin

Graph and locate the foci: $25x^2 + 16y^2 = 400.$

Solution We begin by expressing the equation in standard form. Because we want 1 on the right side, we divide both sides by 400.

$$\frac{25x^2}{400} + \frac{16y^2}{400} = \frac{400}{400}$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

$b^2 = 16$. This is the smaller of the two denominators.

$a^2 = 25$. This is the larger of the two denominators.

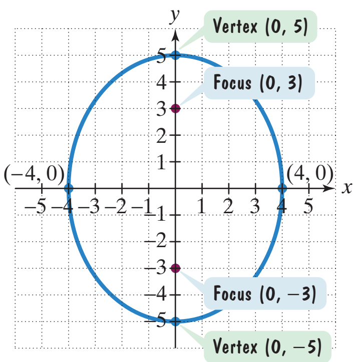


Figure 9.8 The graph of $25x^2 + 16y^2 = 400$, or $\frac{x^2}{16} + \frac{y^2}{25} = 1$

The equation is the standard form of an ellipse's equation with $a^2 = 25$ and $b^2 = 16$. Because the denominator of the y^2-term is greater than the denominator of the x^2-term, the major axis is vertical. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 25$, $a = 5$. Thus, the vertices are $(0, -5)$ and $(0, 5)$, shown in **Figure 9.8**.

Now let us find the endpoints of the horizontal minor axis. According to the standard form of the equation, these endpoints are $(-b, 0)$ and $(b, 0)$. Because

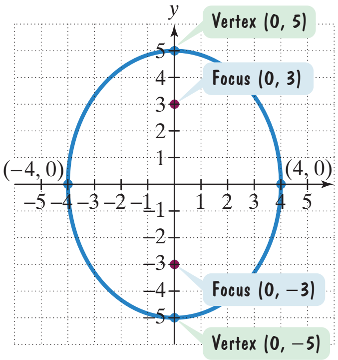


Figure 9.8 (repeated) The graph of $\frac{x^2}{16} + \frac{y^2}{25} = 1$

$b^2 = 16, b = 4$. Thus, the endpoints of the minor axis are $(-4, 0)$ and $(4, 0)$. They are shown in **Figure 9.8**.

Finally, we find the foci, which are located at $(0, -c)$ and $(0, c)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 25$ and $b^2 = 16$. Thus,

$$c^2 = a^2 - b^2 = 25 - 16 = 9.$$

Because $c^2 = 9, c = 3$. The foci, $(0, -c)$ and $(0, c)$, are located at $(0, -3)$ and $(0, 3)$. They are shown in **Figure 9.8**.

You can sketch the ellipse in **Figure 9.8** by locating endpoints on the major and minor axes.

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$

Endpoints of the minor axis are 4 units to the right and left of the center.

Endpoints of the major axis are 5 units up and down from the center.

2 Write equations of ellipses in standard form.

Check Point 2 Graph and locate the foci: $16x^2 + 9y^2 = 144$.

In Examples 1 and 2, we used the equation of an ellipse to find its foci and vertices. In the next example, we reverse this procedure.

EXAMPLE 3 Finding the Equation of an Ellipse from Its Foci and Vertices

Find the standard form of the equation of an ellipse with foci at $(-1, 0)$ and $(1, 0)$ and vertices $(-2, 0)$ and $(2, 0)$.

Solution Because the foci are located at $(-1, 0)$ and $(1, 0)$, on the x-axis, the major axis is horizontal. The center of the ellipse is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, to either vertex, $(-2, 0)$ or $(2, 0)$, is 2. Thus, $a = 2$.

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{b^2} = 1$$

We must still find b^2. The distance from the center, $(0, 0)$, to either focus, $(-1, 0)$ or $(1, 0)$, is 1, so $c = 1$. Using $c^2 = a^2 - b^2$, we have

$$1^2 = 2^2 - b^2$$

and

$$b^2 = 2^2 - 1^2 = 4 - 1 = 3.$$

Substituting 3 for b^2 in $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$ gives us the standard form of the ellipse's equation. The equation is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Check Point 3 Find the standard form of the equation of an ellipse with foci at $(-2, 0)$ and $(2, 0)$ and vertices $(-3, 0)$ and $(3, 0)$.

3 Graph ellipses not centered at the origin.

Translations of Ellipses

Horizontal and vertical translations can be used to graph ellipses that are not centered at the origin. **Figure 9.9** illustrates that the graphs of

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

have the same size and shape. However, the graph of the first equation is centered at (h, k) rather than at the origin.

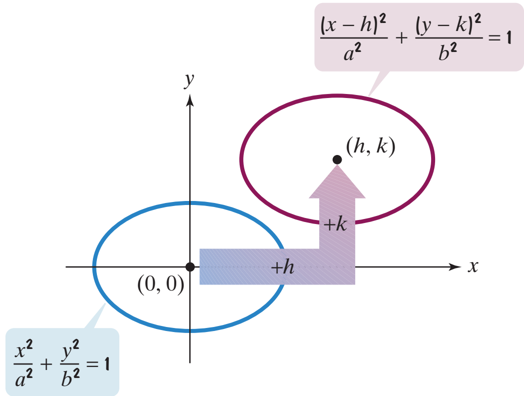
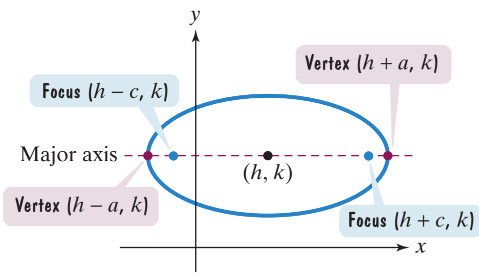
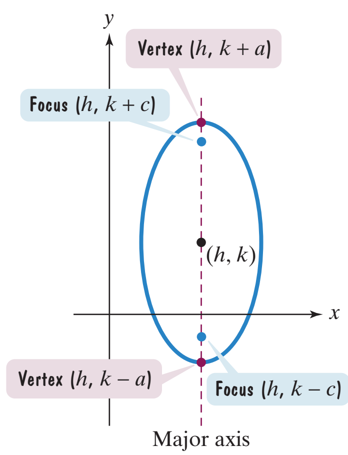


Figure 9.9 Translating an ellipse's graph

Table 9.1 gives the standard forms of equations of ellipses centered at (h, k) and shows their graphs.

Table 9.1 Standard Forms of Equations of Ellipses Centered at (h, k)

Equation	Center	Major Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ Endpoints of major axis are a units right and a units left of center. $a^2 > b^2$ Foci are c units right and c units left of center, where $c^2 = a^2 - b^2$.	(h, k)	Parallel to the x-axis, horizontal	$(h-a, k)$ $(h+a, k)$	
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $a^2 > b^2$ Endpoints of the major axis are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2 = a^2 - b^2$.	(h, k)	Parallel to the y-axis, vertical	$(h, k-a)$ $(h, k+a)$	

EXAMPLE 4 Graphing an Ellipse Centered at (h, k)

Graph: $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$. Where are the foci located?

Solution To graph the ellipse, we need to know its center, (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and k is the number subtracted from y.

This is $(x-h)^2$, with $h = 1$.

This is $(y-k)^2$, with $k = -2$.

$$\frac{(x-1)^2}{4} + \frac{(y-(-2))^2}{9} = 1$$

We see that $h = 1$ and $k = -2$. Thus, the center of the ellipse, (h, k), is $(1, -2)$. We can graph the ellipse by locating endpoints on the major and minor axes. To do this, we must identify a^2 and b^2.

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

$b^2 = 4$. This is the smaller of the two denominators.

$a^2 = 9$. This is the larger of the two denominators.

The larger number is under the expression involving y. This means that the major axis is vertical and parallel to the y-axis.

We can sketch the ellipse by locating endpoints on the major and minor axes.

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} = 1$$

Endpoints of the minor axis are 2 units to the right and left of the center.

Endpoints of the major axis (the vertices) are 3 units up and down from the center.

We categorize the observations in the voice balloons as follows:

For a Vertical Major Axis with Center (1, −2)	
Vertices	**Endpoints of Minor Axis**
$(1, -2 + 3) = (1, 1)$	$(1 + 2, -2) = (3, -2)$
$(1, -2 - 3) = (1, -5)$	$(1 - 2, -2) = (-1, -2)$

3 units above and below center

2 units right and left of center

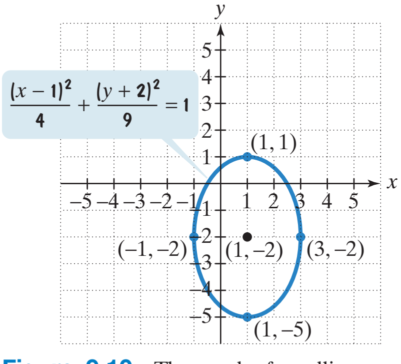


Figure 9.10 The graph of an ellipse centered at $(1, -2)$

Using the center and these four points, we can sketch the ellipse shown in **Figure 9.10**.

With $c^2 = a^2 - b^2$, we have $c^2 = 9 - 4 = 5$. So the foci are located $\sqrt{5}$ units above and below the center, at $(1, -2 + \sqrt{5})$ and $(1, -2 - \sqrt{5})$.

Check Point 4 Graph: $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$. Where are the foci located?

In some cases, it is necessary to convert the equation of an ellipse to standard form by completing the square on x and y. For example, suppose that we wish to graph the ellipse whose equation is

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

Because we plan to complete the square on both x and y, we need to rearrange terms so that

- x-terms are arranged in descending order.
- y-terms are arranged in descending order.
- the constant term appears on the right.

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0$$

This is the given equation.

$$(9x^2 - 18x) + (4y^2 + 16y) = 11$$

Group terms and add 11 to both sides.

$$9(x^2 - 2x + \square) + 4(y^2 + 4y + \square) = 11$$

To complete the square, coefficients of x^2 and y^2 must be 1. Factor out 9 and 4, respectively.

We added $9 \cdot 1$, or 9, to the left side.

We also added $4 \cdot 4$, or 16, to the left side.

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16$$

9 and 16, added on the left side, must also be added on the right side.

Complete each square by adding the square of half the coefficient of x and y, respectively.

$$9(x-1)^2 + 4(y+2)^2 = 36$$

Factor.

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = \frac{36}{36}$$

Divide both sides by 36.

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

Simplify.

The equation is now in standard form. This is precisely the form of the equation that we graphed in Example 4.

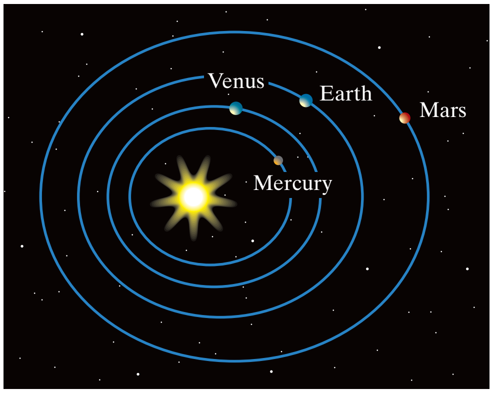
Study Tip

When completing the square, remember that changes made on the left side of the equation must also be made on the right side of the equation.

4 Solve applied problems involving ellipses.

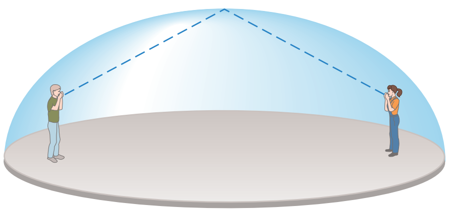
Applications

Ellipses have many applications. German scientist Johannes Kepler (1571–1630) showed that the planets in our solar system move in elliptical orbits, with the sun at a focus. Earth satellites also travel in elliptical orbits, with Earth at a focus.



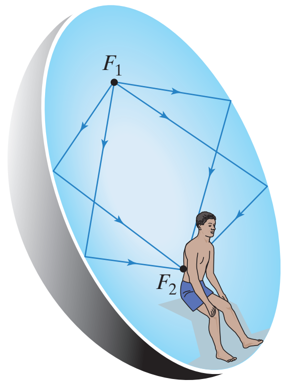
Planets move in elliptical orbits.

One intriguing aspect of the ellipse is that a ray of light or a sound wave emanating from one focus will be reflected from the ellipse to exactly the other focus. A whispering gallery is an elliptical room with an elliptical, dome-shaped ceiling. People standing at the foci can whisper and hear each other quite clearly, while persons in other locations in the room cannot hear them. Statuary Hall in the U.S. Capitol Building is elliptical. President John Quincy Adams, while a member of the House of Representatives, was aware of this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, easily eavesdropping on the private conversations of other House members located near the other focus.



Whispering in an elliptical dome

The elliptical reflection principle is used in a procedure for disintegrating kidney stones. The patient is placed within a device that is elliptical in shape. The patient is placed so the kidney is centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone. The convergence of the ultrasound waves at the kidney stone causes vibrations that shatter it into fragments. The small pieces can then be passed painlessly through the patient's system. The patient recovers in days, as opposed to up to six weeks if surgery is used instead.



Disintegrating kidney stones

Ellipses are often used for supporting arches of bridges and in tunnel construction. This application forms the basis of our next example.

EXAMPLE 5 An Application Involving an Ellipse

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet (see **Figure 9.11**). Your truck has a width of 10 feet and a height of 9 feet. Will your truck clear the opening of the archway?

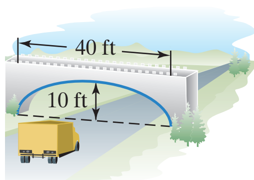


Figure 9.11
A semielliptical archway

Solution Because your truck's width is 10 feet, to determine the clearance, we must find the height of the archway 5 feet from the center. If that height is 9 feet or less, the truck will not clear the opening.

In **Figure 9.12**, we've constructed a coordinate system with the x-axis on the ground and the origin at the center of the archway. Also shown is the truck, whose height is 9 feet.

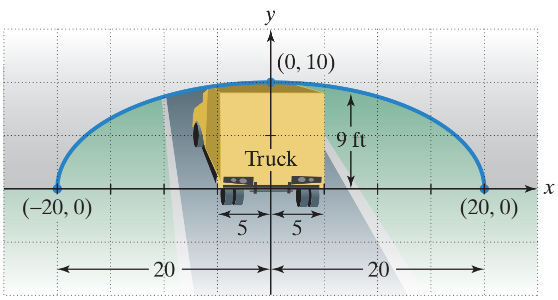
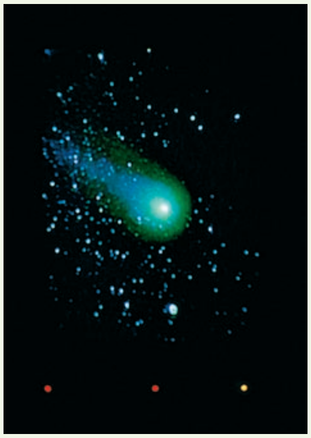


Figure 9.12

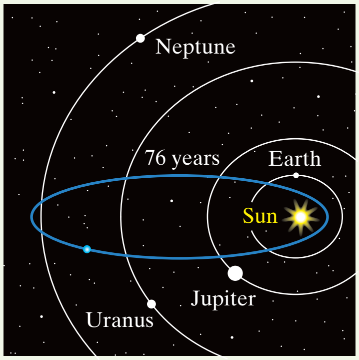
Using the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we can express the equation of the blue archway in **Figure 9.12** as $\frac{x^2}{20^2} + \frac{y^2}{10^2} = 1$, or $\frac{x^2}{400} + \frac{y^2}{100} = 1$.

As shown in **Figure 9.12**, the edge of the 10-foot-wide truck corresponds to $x = 5$. We find the height of the archway 5 feet from the center by substituting 5 for x and solving for y.

Halley's Comet



Halley's Comet has an elliptical orbit with the sun at one focus. The comet returns every 76.3 years. The first recorded sighting was in 239 B.C. It was last seen in 1986. At that time, spacecraft went close to the comet, measuring its nucleus to be 7 miles long and 4 miles wide. By 2024, Halley's Comet will have reached the farthest point in its elliptical orbit before returning to be next visible from Earth in 2062.



The elliptical orbit of Halley's Comet

$$\frac{5^2}{400} + \frac{y^2}{100} = 1 \qquad \text{Substitute 5 for } x \text{ in } \frac{x^2}{400} + \frac{y^2}{100} = 1.$$

$$\frac{25}{400} + \frac{y^2}{100} = 1 \qquad \text{Square 5.}$$

$$400\left(\frac{25}{400} + \frac{y^2}{100}\right) = 400(1) \qquad \text{Clear fractions by multiplying both sides by 400.}$$

$$25 + 4y^2 = 400 \qquad \text{Use the distributive property and simplify.}$$

$$4y^2 = 375 \qquad \text{Subtract 25 from both sides.}$$

$$y^2 = \frac{375}{4} \qquad \text{Divide both sides by 4.}$$

$$y = \sqrt{\frac{375}{4}} \qquad \text{Take only the positive square root. The archway is above the } x\text{-axis, so } y \text{ is nonnegative.}$$

$$\approx 9.68 \qquad \text{Use a calculator.}$$

Thus, the height of the archway 5 feet from the center is approximately 9.68 feet. Because your truck's height is 9 feet, there is enough room for the truck to clear the archway.

Check Point 5 Will a truck that is 12 feet wide and has a height of 9 feet clear the opening of the archway described in Example 5?

Exercise Set 9.1

Practice Exercises

In Exercises 1–18, graph each ellipse and locate the foci.

1. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

2. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

3. $\frac{x^2}{9} + \frac{y^2}{36} = 1$

4. $\frac{x^2}{16} + \frac{y^2}{49} = 1$

5. $\frac{x^2}{25} + \frac{y^2}{64} = 1$

6. $\frac{x^2}{49} + \frac{y^2}{36} = 1$

7. $\frac{x^2}{49} + \frac{y^2}{81} = 1$

8. $\frac{x^2}{64} + \frac{y^2}{100} = 1$

9. $\frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{25}{4}} = 1$

10. $\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{25}{16}} = 1$

11. $x^2 = 1 - 4y^2$

12. $y^2 = 1 - 4x^2$

13. $25x^2 + 4y^2 = 100$

14. $9x^2 + 4y^2 = 36$

15. $4x^2 + 16y^2 = 64$

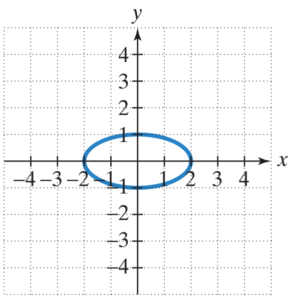
16. $4x^2 + 25y^2 = 100$

17. $7x^2 = 35 - 5y^2$

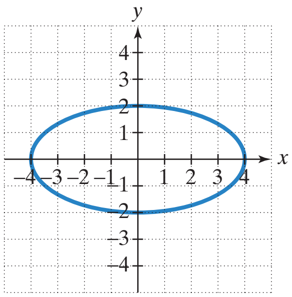
18. $6x^2 = 30 - 5y^2$

In Exercises 19–24, find the standard form of the equation of each ellipse and give the location of its foci.

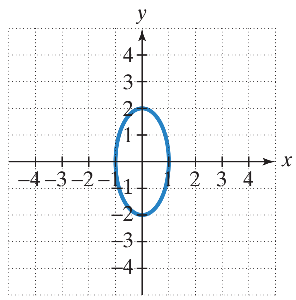
19.



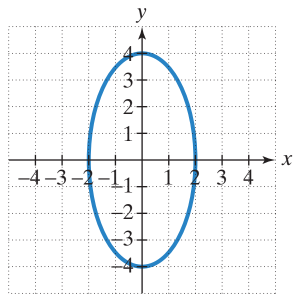
20.



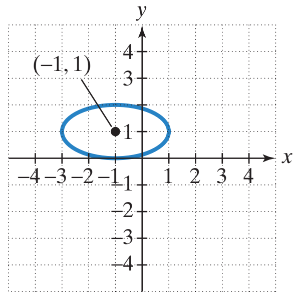
21.



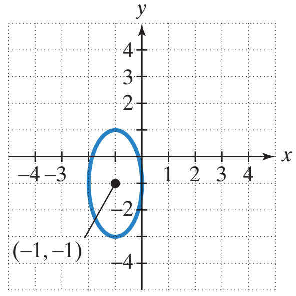
22.



23.



24.



In Exercises 25–36, find the standard form of the equation of each ellipse satisfying the given conditions.

25. Foci: $(-5, 0), (5, 0)$; vertices: $(-8, 0), (8, 0)$

26. Foci: $(-2, 0), (2, 0)$; vertices: $(-6, 0), (6, 0)$

27. Foci: $(0, -4), (0, 4)$; vertices: $(0, -7), (0, 7)$

28. Foci: $(0, -3), (0, 3)$; vertices: $(0, -4), (0, 4)$

29. Foci: $(-2, 0), (2, 0)$; y-intercepts: -3 and 3

30. Foci: $(0, -2), (0, 2)$; x-intercepts: -2 and 2

31. Major axis horizontal with length 8; length of minor axis $= 4$; center: $(0, 0)$

32. Major axis horizontal with length 12; length of minor axis $= 6$; center: $(0, 0)$

33. Major axis vertical with length 10; length of minor axis $= 4$; center: $(-2, 3)$

34. Major axis vertical with length 20; length of minor axis $= 10$; center: $(2, -3)$

35. Endpoints of major axis: $(7, 9)$ and $(7, 3)$
Endpoints of minor axis: $(5, 6)$ and $(9, 6)$

36. Endpoints of major axis: $(2, 2)$ and $(8, 2)$
Endpoints of minor axis: $(5, 3)$ and $(5, 1)$

In Exercises 37–50, graph each ellipse and give the location of its foci.

37. $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

38. $\dfrac{(x-1)^2}{16} + \dfrac{(y+2)^2}{9} = 1$

39. $(x+3)^2 + 4(y-2)^2 = 16$

40. $(x-3)^2 + 9(y+2)^2 = 18$

41. $\dfrac{(x-4)^2}{9} + \dfrac{(y+2)^2}{25} = 1$

42. $\dfrac{(x-3)^2}{9} + \dfrac{(y+1)^2}{16} = 1$

43. $\dfrac{x^2}{25} + \dfrac{(y-2)^2}{36} = 1$

44. $\dfrac{(x-4)^2}{4} + \dfrac{y^2}{25} = 1$

45. $\dfrac{(x+3)^2}{9} + (y-2)^2 = 1$

46. $\dfrac{(x+2)^2}{16} + (y-3)^2 = 1$

47. $\dfrac{(x-1)^2}{2} + \dfrac{(y+3)^2}{5} = 1$

48. $\dfrac{(x+1)^2}{2} + \dfrac{(y-3)^2}{5} = 1$

49. $9(x-1)^2 + 4(y+3)^2 = 36$

50. $36(x+4)^2 + (y+3)^2 = 36$

In Exercises 51–56, convert each equation to standard form by completing the square on x and y. Then graph the ellipse and give the location of its foci.

51. $9x^2 + 25y^2 - 36x + 50y - 164 = 0$

52. $4x^2 + 9y^2 - 32x + 36y + 64 = 0$

53. $9x^2 + 16y^2 - 18x + 64y - 71 = 0$

54. $x^2 + 4y^2 + 10x - 8y + 13 = 0$

55. $4x^2 + y^2 + 16x - 6y - 39 = 0$

56. $4x^2 + 25y^2 - 24x + 100y + 36 = 0$

Practice Plus

In Exercises 57–62, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

57. $\begin{cases} x^2 + y^2 = 1 \\ x^2 + 9y^2 = 9 \end{cases}$

58. $\begin{cases} x^2 + y^2 = 25 \\ 25x^2 + y^2 = 25 \end{cases}$

59. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1 \\ y = 3 \end{cases}$

60. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{36} = 1 \\ x = -2 \end{cases}$

61. $\begin{cases} 4x^2 + y^2 = 4 \\ 2x - y = 2 \end{cases}$

62. $\begin{cases} 4x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$

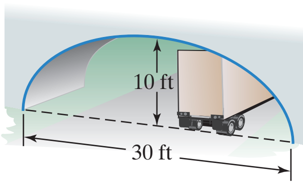
In Exercises 63–64, graph each semiellipse.

63. $y = -\sqrt{16 - 4x^2}$

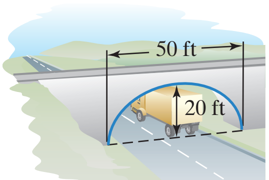
64. $y = -\sqrt{4 - 4x^2}$

Application Exercises

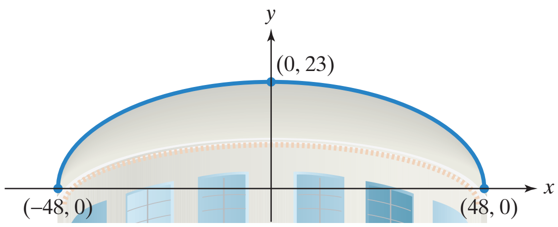
65. Will a truck that is 8 feet wide carrying a load that reaches 7 feet above the ground clear the semielliptical arch on the one-way road that passes under the bridge shown in the figure?



66. A semielliptic archway has a height of 20 feet and a width of 50 feet, as shown in the figure. Can a truck 14 feet high and 10 feet wide drive under the archway without going into the other lane?



67. The elliptical ceiling in Statuary Hall in the U.S. Capitol Building is 96 feet long and 23 feet tall.



a. Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling.

b. John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk? (Round to the nearest foot.)

68. If an elliptical whispering room has a height of 30 feet and a width of 100 feet, where should two people stand if they would like to whisper back and forth and be heard?

Writing in Mathematics

69. What is an ellipse?

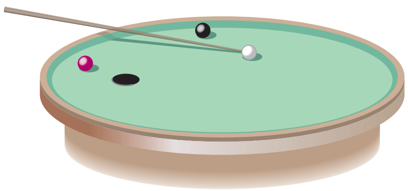
70. Describe how to graph $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

71. Describe how to locate the foci for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

72. Describe one similarity and one difference between the graphs of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

73. Describe one similarity and one difference between the graphs of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $\frac{(x-1)^2}{25} + \frac{(y-1)^2}{16} = 1$.

74. An elliptipool is an elliptical pool table with only one pocket. A pool shark places a ball on the table, hits it in what appears to be a random direction, and yet it bounces off the edge, falling directly into the pocket. Explain why this happens.



Technology Exercises

75. Use a graphing utility to graph any five of the ellipses that you graphed by hand in Exercises 1–18.

76. Use a graphing utility to graph any three of the ellipses that you graphed by hand in Exercises 37–50. First solve the given equation for y by using the square root property. Enter each of the two resulting equations to produce each half of the ellipse.

77. Use a graphing utility to graph any one of the ellipses that you graphed by hand in Exercises 51–56. Write the equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the two resulting equations to produce each half of the ellipse.

78. Write an equation for the path of each of the following elliptical orbits. Then use a graphing utility to graph the two ellipses in the same viewing rectangle. Can you see why early astronomers had difficulty detecting that these orbits are ellipses rather than circles?

- Earth's orbit: Length of major axis: 186 million miles
  Length of minor axis: 185.8 million miles
- Mars's orbit: Length of major axis: 283.5 million miles
  Length of minor axis: 278.5 million miles

Critical Thinking Exercises

Make Sense? *In Exercises 79–82, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

79. I graphed an ellipse with a horizontal major axis and foci on the y-axis.

80. I graphed an ellipse that was symmetric about its major axis but not symmetric about its minor axis.

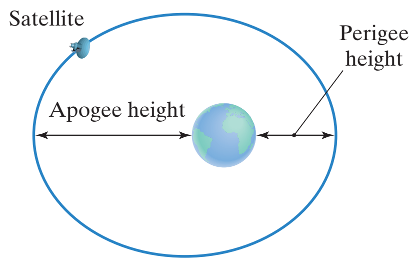
81. You told me that an ellipse centered at the origin has vertices at $(-5, 0)$ and $(5, 0)$, so I was able to graph the ellipse.

82. In a whispering gallery at our science museum, I stood at one focus, my friend stood at the other focus, and we had a clear conversation, very little of which was heard by the 25 museum visitors standing between us.

83. Find the standard form of the equation of an ellipse with vertices at $(0, -6)$ and $(0, 6)$, passing through $(2, -4)$.

84. An Earth satellite has an elliptical orbit described by

$$\frac{x^2}{(5000)^2} + \frac{y^2}{(4750)^2} = 1.$$



(All units are in miles.) The coordinates of the center of Earth are $(16, 0)$.

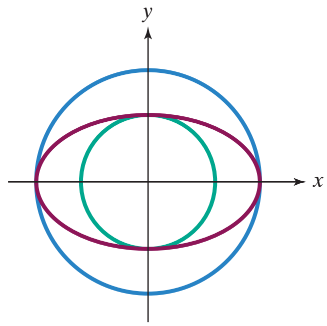
a. The perigee of the satellite's orbit is the point that is nearest Earth's center. If the radius of Earth is approximately 4000 miles, find the distance of the perigee above Earth's surface.

b. The apogee of the satellite's orbit is the point that is the greatest distance from Earth's center. Find the distance of the apogee above Earth's surface.

85. The equation of the red ellipse in the figure shown is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Write the equation for each circle shown in the figure.



86. What happens to the shape of the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as $\frac{c}{a} \to 0$, where $c^2 = a^2 - b^2$?

Preview Exercises

Exercises 87–89 will help you prepare for the material covered in the next section.

87. Divide both sides of $4x^2 - 9y^2 = 36$ by 36 and simplify. How does the simplified equation differ from that of an ellipse?

88. Consider the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

a. Find the x-intercepts.

b. Explain why there are no y-intercepts.

89. Consider the equation $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

a. Find the y-intercepts.

b. Explain why there are no x-intercepts.