Group Exercise

69. Many public and private organizations and schools provide educational materials and information for the blind and visually impaired. Using your library, resources on the World Wide Web, or local organizations, investigate how your group or college could make a contribution to enhance the study of mathematics for the blind and visually impaired. In relation to conic sections, group members should discuss how to create graphs in tactile, or touchable, form that show blind students the visual structure of the conics, including asymptotes, intercepts, end behavior, and rotations.

Preview Exercises

Exercises 70–72 will help you prepare for the material covered in the next section. In each exercise, graph the equation in a rectangular coordinate system.

70. $y^2 = 4(x + 1)$

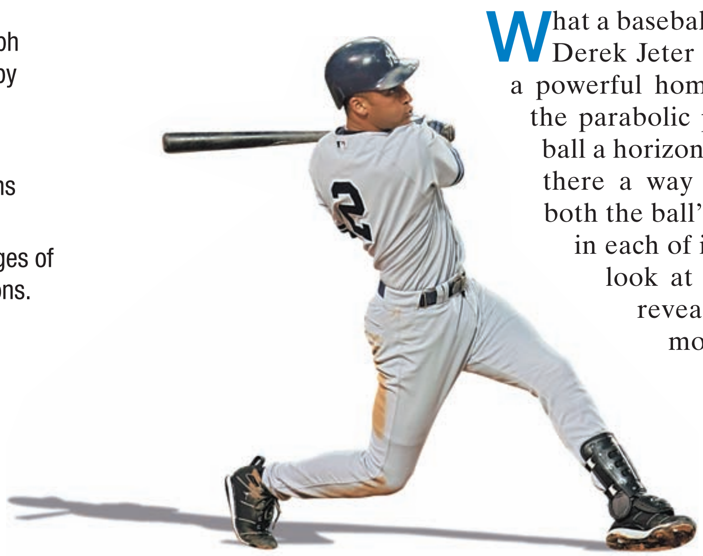
71. $y = \frac{1}{2}x^2 + 1, \quad x \geq 0$

72. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

Section 9.5 Parametric Equations

Objectives

1. Use point plotting to graph plane curves described by parametric equations.
2. Eliminate the parameter.
3. Find parametric equations for functions.
4. Understand the advantages of parametric representations.



What a baseball game! You got to see the great Derek Jeter of the New York Yankees blast a powerful homer. In less than eight seconds, the parabolic path of his home run took the ball a horizontal distance of over 1000 feet. Is there a way to model this path that gives both the ball's location and the time that it is in each of its positions? In this section, we look at ways of describing curves that reveal the where and the when of motion.

Plane Curves and Parametric Equations

You throw a ball from a height of 6 feet, with an initial velocity of 90 feet per second and at an angle of 40° with the horizontal. After t seconds, the location of the ball can be described by

$$x = (90 \cos 40°)t \quad \text{and} \quad y = 6 + (90 \sin 40°)t - 16t^2.$$

This is the ball's horizontal distance, in feet.

This is the ball's vertical height, in feet.

Because we can use these equations to calculate the location of the ball at any time t, we can describe the path of the ball. For example, to determine the location when $t = 1$ second, substitute 1 for t in each equation:

$$x = (90 \cos 40°)t = (90 \cos 40°)(1) \approx 68.9 \text{ feet}$$

$$y = 6 + (90 \sin 40°)t - 16t^2 = 6 + (90 \sin 40°)(1) - 16(1)^2 \approx 47.9 \text{ feet.}$$

This tells us that after one second, the ball has traveled a horizontal distance of approximately 68.9 feet, and the height of the ball is approximately 47.9 feet. **Figure 9.49** on the next page displays this information and the results for calculations corresponding to $t = 2$ seconds and $t = 3$ seconds.

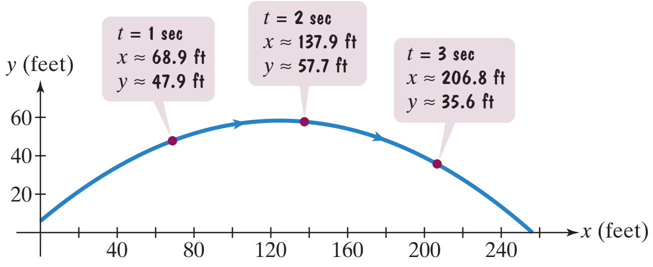


Figure 9.49 The location of a thrown ball after 1, 2, and 3 seconds

The voice balloons in **Figure 9.49** tell where the ball is located and when the ball is at a given point (x, y) on its path. The variable t, called a **parameter**, gives the various times for the ball's location. The equations that describe where the ball is located express both x and y as functions of t and are called **parametric equations**.

$$x = (90 \cos 40°)t \qquad y = 6 + (90 \sin 40°)t - 16t^2$$

This is the parametric equation for x.

This is the parametric equation for y.

The collection of points (x, y) in **Figure 9.49** is called a **plane curve**.

Plane Curves and Parametric Equations

Suppose that t is a number in an interval I. A **plane curve** is the set of ordered pairs (x, y), where

$$x = f(t), \quad y = g(t) \quad \text{for } t \text{ in interval } I.$$

The variable t is called a **parameter**, and the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** for the curve.

1 Use point plotting to graph plane curves described by parametric equations.

Graphing Plane Curves

Graphing a plane curve represented by parametric equations involves plotting points in the rectangular coordinate system and connecting them with a smooth curve.

Graphing a Plane Curve Described by Parametric Equations

1. Select some values of t on the given interval.
2. For each value of t, use the given parametric equations to compute x and y.
3. Plot the points (x, y) in the order of increasing t and connect them with a smooth curve.

Take a second look at **Figure 9.49**. Do you notice arrows along the curve? These arrows show the direction, or **orientation**, along the curve as t increases. After graphing a plane curve described by parametric equations, use arrows between the points to show the orientation of the curve corresponding to increasing values of t.

EXAMPLE 1 Graphing a Curve Defined by Parametric Equations

Graph the plane curve defined by the parametric equations:

$$x = t^2 - 1, \qquad y = 2t, \qquad -2 \le t \le 2.$$

Solution

Step 1 Select some values of t on the given interval. We will select integral values of t on the interval $-2 \le t \le 2$. Let $t = -2, -1, 0, 1,$ and 2.

Step 2 For each value of t, use the given parametric equations to compute x and y. We organize our work in a table. The first column lists the choices for the parameter t. The next two columns show the corresponding values for x and y. The last column lists the ordered pair (x, y).

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-2	$(-2)^2 - 1 = 4 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 1 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$0^2 - 1 = -1$	$2(0) = 0$	$(-1, 0)$
1	$1^2 - 1 = 0$	$2(1) = 2$	$(0, 2)$
2	$2^2 - 1 = 4 - 1 = 3$	$2(2) = 4$	$(3, 4)$

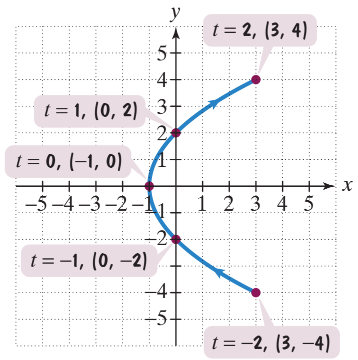


Figure 9.50 The plane curve defined by $x = t^2 - 1$, $y = 2t$, $-2 \le t \le 2$

Step 3 Plot the points (x, y) in the order of increasing t and connect them with a smooth curve. The plane curve defined by the parametric equations on the given interval is shown in **Figure 9.50**. The arrows show the direction, or orientation, along the curve as t varies from -2 to 2.

Check Point 1 Graph the plane curve defined by the parametric equations:

$$x = t^2 + 1, \quad y = 3t, \quad -2 \le t \le 2.$$

2 Eliminate the parameter.

Eliminating the Parameter

The graph in **Figure 9.50** shows the plane curve for $x = t^2 - 1$, $y = 2t$, $-2 \le t \le 2$. Even if we examine the parametric equations carefully, we may not be able to tell that the corresponding plane curve is a portion of a parabola. By **eliminating the parameter**, we can write one equation in x and y that is equivalent to the two parametric equations. The voice balloons illustrate this process.

Begin with the parametric equations.

$$x = t^2 - 1$$
$$y = 2t$$

Solve for t in one of the equations.

Using $y = 2t$,

$$t = \frac{y}{2}.$$

Substitute the expression for t in the other parametric equation.

Using $t = \frac{y}{2}$ and $x = t^2 - 1$,

$$x = \left(\frac{y}{2}\right)^2 - 1.$$

The rectangular equation (the equation in x and y), $x = \frac{y^2}{4} - 1$, can be written as $y^2 = 4(x + 1)$. This is the standard form of the equation of a parabola with vertex at $(-1, 0)$ and axis of symmetry along the x-axis. Because the parameter t is restricted to the interval $[-2, 2]$, the plane curve in the technology box on the left shows only a part of the parabola.

Our discussion illustrates a second method for graphing a plane curve described by parametric equations. Eliminate the parameter t and graph the resulting rectangular equation in x and y. However, **you may need to change the domain of the rectangular equation to be consistent with the domain for the parametric equation in x.** This situation is illustrated in Example 2.

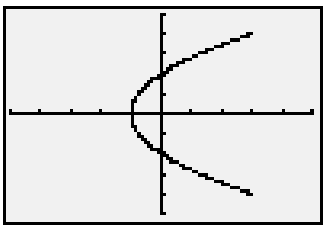
Technology

A graphing utility can be used to obtain a plane curve represented by parametric equations. Set the mode to parametric and enter the equations. You must enter the minimum and maximum values for t, and an increment setting for t ($t$step). The setting $t$step determines the number of points the graphing utility will plot.

Shown below is the plane curve for

$$x = t^2 - 1$$
$$y = 2t$$

in a $[-5, 5, 1]$ by $[-5, 5, 1]$ viewing rectangle with $t$min $= -2$, $t$max $= 2$, and $t$step $= 0.01$.



EXAMPLE 2 Finding and Graphing the Rectangular Equation of a Curve Defined Parametrically

Sketch the plane curve represented by the parametric equations

$$x = \sqrt{t} \quad \text{and} \quad y = \tfrac{1}{2}t + 1$$

by eliminating the parameter.

Solution We eliminate the parameter t and then graph the resulting rectangular equation.

Begin with the parametric equations.	Solve for t in one of the other equations.	Substitute the expression for t in the other parametric equation.
$x = \sqrt{t}$ $y = \frac{1}{2}t + 1$	Using $x = \sqrt{t}$ and squaring both sides, $t = x^2$.	Using $t = x^2$ and $y = \frac{1}{2}t + 1$, $y = \frac{1}{2}x^2 + 1$.

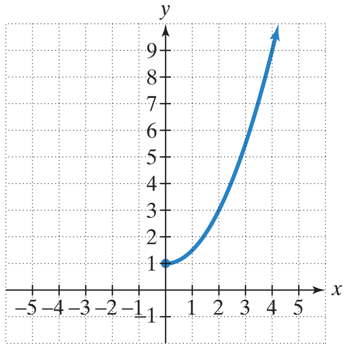


Figure 9.51 The plane curve for $x = \sqrt{t}$ and $y = \frac{1}{2}t + 1$, or $y = \frac{1}{2}x^2 + 1, x \ge 0$

Because t is not limited to a closed interval, you might be tempted to graph the entire bowl-shaped parabola whose equation is $y = \frac{1}{2}x^2 + 1$. However, take a second look at the parametric equation for x:

$$x = \sqrt{t}.$$

This equation is defined only when $t \ge 0$. Thus, x is nonnegative. The plane curve is the parabola given by $y = \frac{1}{2}x^2 + 1$ with the domain restricted to $x \ge 0$. The plane curve is shown in **Figure 9.51**.

Check Point 2 Sketch the plane curve represented by the parametric equations

$$x = \sqrt{t} \quad \text{and} \quad y = 2t - 1$$

by eliminating the parameter.

Eliminating the parameter is not always a simple matter. In some cases, it may not be possible. When this occurs, you can use point plotting to obtain a plane curve.

Trigonometric identities can be helpful in eliminating the parameter. For example, consider the plane curve defined by the parametric equations

$$x = \sin t, \quad y = \cos t, \quad 0 \le t < 2\pi.$$

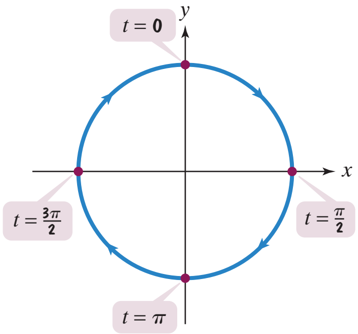


Figure 9.52 The plane curve defined by $x = \sin t$, $y = \cos t$, $0 \le t < 2\pi$

We use the trigonometric identity $\sin^2 t + \cos^2 t = 1$ to eliminate the parameter. Square each side of each parametric equation and then add.

$$\begin{aligned} x^2 &= \sin^2 t \\ y^2 &= \cos^2 t \\ \hline x^2 + y^2 &= \sin^2 t + \cos^2 t \end{aligned}$$

This is the sum of the two equations above the horizontal lines.

Using a Pythagorean identity, we write this equation as $x^2 + y^2 = 1$. The plane curve is a circle with center $(0, 0)$ and radius 1. It is shown in **Figure 9.52**.

EXAMPLE 3 Finding and Graphing the Rectangular Equation of a Curve Defined Parametrically

Sketch the plane curve represented by the parametric equations

$$x = 5\cos t, \quad y = 2\sin t, \quad 0 \le t \le \pi$$

by eliminating the parameter.

Solution We eliminate the parameter using the identity $\cos^2 t + \sin^2 t = 1$. To apply the identity, divide the parametric equation for x by 5 and the parametric equation for y by 2.

$$\frac{x}{5} = \cos t \quad \text{and} \quad \frac{y}{2} = \sin t$$

Square and add these two equations.

$$\frac{x^2}{25} = \cos^2 t$$

$$\frac{y^2}{4} = \sin^2 t$$

$$\frac{x^2}{25} + \frac{y^2}{4} = \cos^2 t + \sin^2 t$$

This is the sum of the two equations above the horizontal lines.

Using a Pythagorean identity, we write this equation as

$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

This rectangular equation is the standard form of the equation for an ellipse centered at $(0, 0)$.

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

$a^2 = 25$: Endpoints of major axis are 5 units left and right of center.

$b^2 = 4$: Endpoints of minor axis are 2 units above and below center.

The ellipse is shown in **Figure 9.53(a)**. However, this is not the plane curve. Because t is restricted to the interval $[0, \pi]$, the plane curve is only a portion of the ellipse. Use the starting and ending values for t, 0 and π, respectively, and a value of t in the interval $(0, \pi)$ to find which portion to include.

Begin at $t = 0$.

$$x = 5\cos t = 5\cos 0 = 5 \cdot 1 = 5$$

$$y = 2\sin t = 2\sin 0 = 2 \cdot 0 = 0$$

Increase to $t = \frac{\pi}{2}$.

$$x = 5\cos t = 5\cos\frac{\pi}{2} = 5 \cdot 0 = 0$$

$$y = 2\sin t = 2\sin\frac{\pi}{2} = 2 \cdot 1 = 2$$

End at $t = \pi$.

$$x = 5\cos t = 5\cos \pi = 5(-1) = -5$$

$$y = 2\sin t = 2\sin \pi = 2(0) = 0$$

Points on the plane curve include $(5, 0)$, which is the starting point, $(0, 2)$, and $(-5, 0)$, which is the ending point. The plane curve is the top half of the ellipse, shown in **Figure 9.53(b)**.

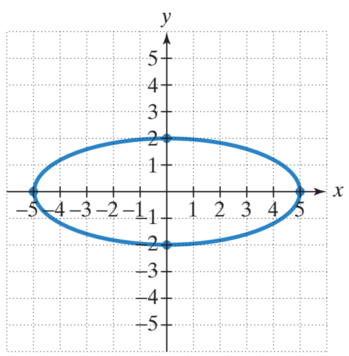


Figure 9.53(a) The graph of $\frac{x^2}{25} + \frac{y^2}{4} = 1$

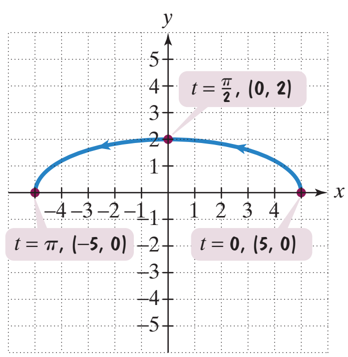


Figure 9.53(b) The plane curve for $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le \pi$

Check Point 3 Sketch the plane curve represented by the parametric equations

$$x = 6\cos t,\ y = 4\sin t,\ \pi \le t \le 2\pi$$

by eliminating the parameter.

3 Find parametric equations for functions.

Finding Parametric Equations

Infinitely many pairs of parametric equations can represent the same plane curve. If the plane curve is defined by the function $y = f(x)$, here is a procedure for finding a set of parametric equations:

Parametric Equations for the Function $y = f(x)$

One set of parametric equations for the plane curve defined by $y = f(x)$ is

$$x = t \quad \text{and} \quad y = f(t),$$

in which t is in the domain of f.

EXAMPLE 4 Finding Parametric Equations

Find a set of parametric equations for the parabola whose equation is $y = 9 - x^2$.

Solution Let $x = t$. Parametric equations for $y = f(x)$ are $x = t$ and $y = f(t)$. Thus, parametric equations for $y = 9 - x^2$ are

$$x = t \quad \text{and} \quad y = 9 - t^2.$$

Check Point 4 Find a set of parametric equations for the parabola whose equation is $y = x^2 - 25$.

You can write other sets of parametric equations for $y = 9 - x^2$ by starting with a different parametric equation for x. Here are three more sets of parametric equations for

$$y = 9 - x^2:$$

- If $x = t^3$, $y = 9 - (t^3)^2 = 9 - t^6$.

  Parametric equations are $x = t^3$ and $y = 9 - t^6$.
- If $x = t + 1$, $y = 9 - (t + 1)^2 = 9 - (t^2 + 2t + 1) = 8 - t^2 - 2t$.

  Parametric equations are $x = t + 1$ and $y = 8 - t^2 - 2t$.
- If $x = \frac{t}{2}$, $y = 9 - \left(\frac{t}{2}\right)^2 = 9 - \frac{t^2}{4}$.

  Parametric equations are $x = \frac{t}{2}$ and $y = 9 - \frac{t^2}{4}$.

Can you start with any choice for the parametric equation for x? The answer is no. **The substitution for x must be a function that allows x to take on all the values in the domain of the given rectangular equation.** For example, the domain of the function $y = 9 - x^2$ is the set of all real numbers. If you incorrectly let $x = t^2$, these values of x exclude negative numbers that are included in $y = 9 - x^2$. The parametric equations

$$x = t^2 \quad \text{and} \quad y = 9 - (t^2)^2 = 9 - t^4$$

do not represent $y = 9 - x^2$ because only points for which $x \geq 0$ are obtained.

4 Understand the advantages of parametric representations.

Advantages of Parametric Equations over Rectangular Equations

We opened this section with parametric equations that described the horizontal distance and the vertical height of your thrown baseball after t seconds. Parametric equations are frequently used to represent the path of a moving object. If t

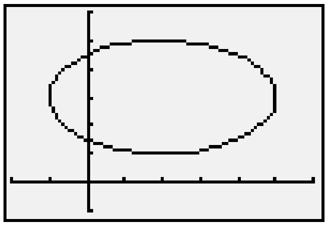
Technology

The ellipse shown was obtained using the parametric mode and the radian mode of a graphing utility.

$$x(t) = 2 + 3\cos t$$

$$y(t) = 3 + 2\sin t$$

We used a $[-2, 6, 1]$ by $[-1, 6, 1]$ viewing rectangle with $t\text{min} = 0$, $t\text{max} = 6.2$, and $t\text{step} = 0.1$.



represents time, parametric equations give the location of a moving object and tell when the object is located at each of its positions. Rectangular equations tell where the moving object is located but do not reveal when the object is in a particular position.

When using technology to obtain graphs, parametric equations that represent relations that are not functions are often easier to use than their corresponding rectangular equations. It is far easier to enter the equation of an ellipse given by the parametric equations

$$x = 2 + 3\cos t \quad \text{and} \quad y = 3 + 2\sin t$$

than to use the rectangular equivalent

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1.$$

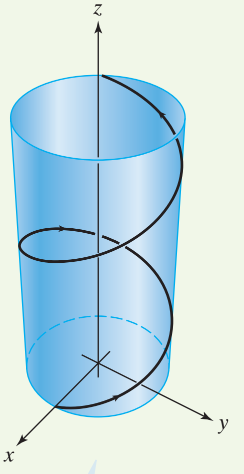
The rectangular equation must first be solved for y and then entered as two separate equations before a graphing utility reveals the ellipse.

The Parametrization of DNA

DNA, the molecule of biological inheritance, is hip. At least that's what a new breed of marketers would like you to believe. For $2500, you can spit into a test tube and a Web-based company will tell you your risks for heart attack and other conditions.

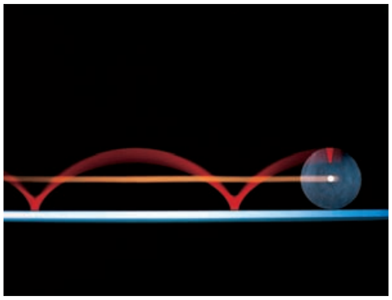
It's been more than 55 years since James Watson and Francis Crick defined the structure, or shape, of DNA. A knowledge of how a molecule is structured does not always lead to an understanding of how it works, but it did in the case of DNA. The structure, which Watson and Crick announced in *Nature* in 1953, immediately suggested how the molecule could be reproduced and how it could contain biological information.

The DNA molecule, structured like a spiraled ladder, consists of two parallel helices (singular: helix) that are intertwined.



Each helix can be described by a curve in three dimensions represented by parametric equations in x, y, and z: $x = a\cos t$, $y = a\sin t$, $z = bt$, where a and b are positive constants.

The structure of the DNA molecule reveals the vital role that trigonometric functions play in the genetic information and instruction codes necessary for the maintenance and continuation of life.



Linear functions and cycloids are used to describe rolling motion. The light at the rolling circle's center shows that it moves linearly. By contrast, the light at the circle's edge has rotational motion and traces out a cycloid.

A curve that is used in physics for much of the theory of light is called a **cycloid**. The path of a fixed point on the circumference of a circle as it rolls along a line is a cycloid. A point on the rim of a bicycle wheel traces out a cycloid curve, shown in **Figure 9.54**. If the radius of the circle is a, the parametric equations of the cycloid are

$$x = a(t - \sin t) \quad \text{and} \quad y = a(1 - \cos t).$$

It is an extremely complicated task to represent the cycloid in rectangular form.

Cycloids are used to solve problems that involve the "shortest time." For example, **Figure 9.55** shows a bead sliding down a wire. For the bead to travel along the wire in the shortest possible time, the shape of the wire should be that of an inverted cycloid.

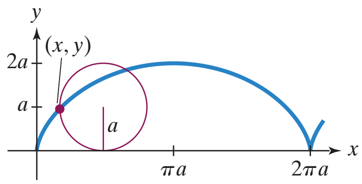


Figure 9.54 The curve traced by a fixed point on the circumference of a circle rolling along a straight line is a cycloid.

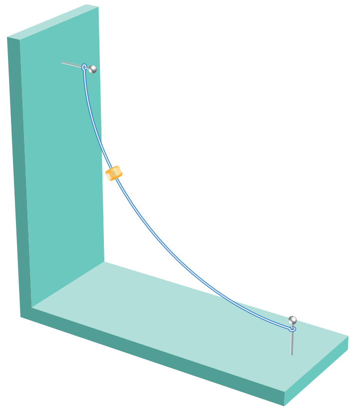


Figure 9.55

Exercise Set 9.5

Practice Exercises

In Exercises 1–8, parametric equations and a value for the parameter t are given. Find the coordinates of the point on the plane curve described by the parametric equations corresponding to the given value of t.

1. $x = 3 - 5t, y = 4 + 2t; t = 1$

2. $x = 7 - 4t, y = 5 + 6t; t = 1$

3. $x = t^2 + 1, y = 5 - t^3; t = 2$

4. $x = t^2 + 3, y = 6 - t^3; t = 2$

5. $x = 4 + 2\cos t, y = 3 + 5\sin t; t = \frac{\pi}{2}$

6. $x = 2 + 3\cos t, y = 4 + 2\sin t; t = \pi$

7. $x = (60\cos 30°)t, y = 5 + (60\sin 30°)t - 16t^2; t = 2$

8. $x = (80\cos 45°)t, y = 6 + (80\sin 45°)t - 16t^2; t = 2$

In Exercises 9–20, use point plotting to graph the plane curve described by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t.

9. $x = t + 2, y = t^2; -2 \le t \le 2$

10. $x = t - 1, y = t^2; -2 \le t \le 2$

11. $x = t - 2, y = 2t + 1; -2 \le t \le 3$

12. $x = t - 3, y = 2t + 2; -2 \le t \le 3$

13. $x = t + 1, y = \sqrt{t}; t \ge 0$

14. $x = \sqrt{t}, y = t - 1; t \ge 0$

15. $x = \cos t, y = \sin t; 0 \le t < 2\pi$

16. $x = -\sin t, y = -\cos t; 0 \le t < 2\pi$

17. $x = t^2, y = t^3; -\infty < t < \infty$

18. $x = t^2 + 1, y = t^3 - 1; -\infty < t < \infty$

19. $x = 2t, y = |t - 1|; -\infty < t < \infty$

20. $x = |t + 1|, y = t - 2; -\infty < t < \infty$

In Exercises 21–40, eliminate the parameter t. Then use the rectangular equation to sketch the plane curve represented by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t. (If an interval for t is not specified, assume that $-\infty < t < \infty$.)

21. $x = t, y = 2t$

22. $x = t, y = -2t$

23. $x = 2t - 4, y = 4t^2$

24. $x = t - 2, y = t^2$

25. $x = \sqrt{t}, y = t - 1$

26. $x = \sqrt{t}, y = t + 1$

27. $x = 2\sin t, y = 2\cos t; 0 \le t < 2\pi$

28. $x = 3\sin t, y = 3\cos t; 0 \le t < 2\pi$

29. $x = 1 + 3\cos t, y = 2 + 3\sin t; 0 \le t < 2\pi$

30. $x = -1 + 2\cos t, y = 1 + 2\sin t; 0 \le t < 2\pi$

31. $x = 2\cos t, y = 3\sin t; 0 \le t < 2\pi$

32. $x = 3\cos t, y = 5\sin t; 0 \le t < 2\pi$

33. $x = 1 + 3\cos t, y = -1 + 2\sin t; 0 \le t \le \pi$

34. $x = 2 + 4\cos t, y = -1 + 3\sin t; 0 \le t \le \pi$

35. $x = \sec t, y = \tan t$

36. $x = 5\sec t, y = 3\tan t$

37. $x = t^2 + 2, y = t^2 - 2$

38. $x = \sqrt{t} + 2, y = \sqrt{t} - 2$

39. $x = 2^t, y = 2^{-t}; t \ge 0$

40. $x = e^t, y = e^{-t}; t \ge 0$

In Exercises 41–43, eliminate the parameter. Write the resulting equation in standard form.

41. A circle: $x = h + r\cos t, y = k + r\sin t$

42. An ellipse: $x = h + a\cos t, y = k + b\sin t$

43. A hyperbola: $x = h + a\sec t, y = k + b\tan t$

44. The following are parametric equations of the line through (x_1, y_1) and (x_2, y_2):

$$x = x_1 + t(x_2 - x_1) \quad \text{and} \quad y = y_1 + t(y_2 - y_1).$$

Eliminate the parameter and write the resulting equation in point-slope form.

In Exercises 45–52, use your answers from Exercises 41–44 and the parametric equations given in Exercises 41–44 to find a set of parametric equations for the conic section or the line.

45. Circle: Center: $(3, 5)$; Radius: 6

46. Circle: Center: $(4, 6)$; Radius: 9

47. Ellipse: Center: $(-2, 3)$; Vertices: 5 units to the left and right of the center; Endpoints of Minor Axis: 2 units above and below the center

48. Ellipse: Center: $(4, -1)$; Vertices: 5 units above and below the center; Endpoints of Minor Axis: 3 units to the left and right of the center

49. Hyperbola: Vertices: $(4, 0)$ and $(-4, 0)$; Foci: $(6, 0)$ and $(-6, 0)$

50. Hyperbola: Vertices: $(0, 4)$ and $(0, -4)$; Foci: $(0, 5)$ and $(0, -5)$

51. Line: Passes through $(-2, 4)$ and $(1, 7)$

52. Line: Passes through $(3, -1)$ and $(9, 12)$

In Exercises 53–56, find two different sets of parametric equations for each rectangular equation.

53. $y = 4x - 3$

54. $y = 2x - 5$

55. $y = x^2 + 4$

56. $y = x^2 - 3$

In Exercises 57–58, the parametric equations of four plane curves are given. Graph each plane curve and determine how they differ from each other.

57. a. $x = t$ and $y = t^2 - 4$

b. $x = t^2$ and $y = t^4 - 4$

c. $x = \cos t$ and $y = \cos^2 t - 4$

d. $x = e^t$ and $y = e^{2t} - 4$

58. a. $x = t, y = \sqrt{4 - t^2}; -2 \le t \le 2$

b. $x = \sqrt{4 - t^2}, y = t; -2 \le t \le 2$

c. $x = 2 \sin t, y = 2 \cos t; 0 \le t < 2\pi$

d. $x = 2 \cos t, y = 2 \sin t; 0 \le t < 2\pi$

Practice Plus

In Exercises 59–62, sketch the plane curve represented by the given parametric equations. Then use interval notation to give each relation's domain and range.

59. $x = 4 \cos t + 2, y = 4 \cos t - 1$

60. $x = 2 \sin t - 3, y = 2 \sin t + 1$

61. $x = t^2 + t + 1, y = 2t$

62. $x = t^2 - t + 6, y = 3t$

In Exercises 63–68, sketch the function represented by the given parametric equations. Then use the graph to determine each of the following:

a. *intervals, if any, on which the function is increasing and intervals, if any, on which the function is decreasing.*

b. *the number, if any, at which the function has a maximum and this maximum value, or the number, if any, at which the function has a minimum and this minimum value.*

63. $x = 2^t, y = t$

64. $x = e^t, y = t$

65. $x = \frac{t}{2}, y = 2t^2 - 8t + 3$

66. $x = \frac{t}{2}, y = -2t^2 + 8t - 1$

67. $x = 2(t - \sin t), y = 2(1 - \cos t); 0 \le t \le 2\pi$

68. $x = 3(t - \sin t), y = 3(1 - \cos t); 0 \le t \le 2\pi$

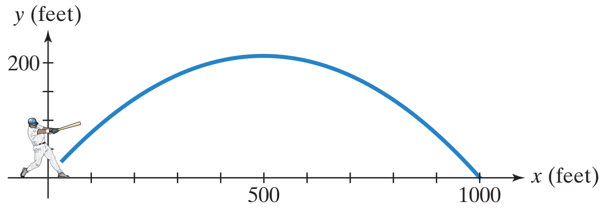
Application Exercises

The path of a projectile that is launched h feet above the ground with an initial velocity of v_0 feet per second and at an angle θ with the horizontal is given by the parametric equations

$$x = (v_0 \cos \theta)t \quad \text{and} \quad y = h + (v_0 \sin \theta)t - 16t^2,$$

where t is the time, in seconds, after the projectile was launched. The parametric equation for x gives the projectile's horizontal distance, in feet. The parametric equation for y gives the projectile's height, in feet. Use these parametric equations to solve Exercises 69–70.

69. The figure shows the path for a baseball hit by Derek Jeter. The ball was hit with an initial velocity of 180 feet per second at an angle of 40° to the horizontal. The ball was hit at a height 3 feet off the ground.



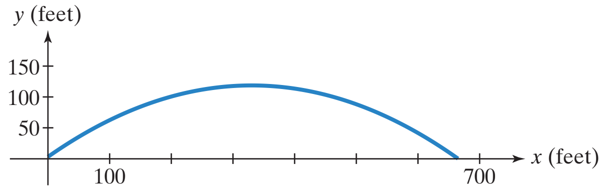
a. Find the parametric equations that describe the position of the ball as a function of time.

b. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot. Locate your solutions on the plane curve.

c. How long, to the nearest tenth of a second, is the ball in flight? What is the total horizontal distance that it travels before it lands? Is your answer consistent with the figure shown?

d. You meet Derek Jeter and he asks you to tell him something interesting about the path of the baseball that he hit. Use the graph to respond to his request. Then verify your observation algebraically.

70. The figure shows the path for a baseball that was hit with an initial velocity of 150 feet per second at an angle of 35° to the horizontal. The ball was hit at a height of 3 feet off the ground.



a. Find the parametric equations that describe the position of the ball as a function of time.

b. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot. Locate your solutions on the plane curve.

c. How long is the ball in flight? (Round to the nearest tenth of a second.) What is the total horizontal distance that it travels, to the nearest tenth of a foot, before it lands? Is your answer consistent with the figure shown at the bottom of the previous page?

d. Use the graph on the previous page to describe something about the path of the baseball that might be of interest to the player who hit the ball. Then verify your observation algebraically.

Writing in Mathematics

71. What are plane curves and parametric equations?

72. How is point plotting used to graph a plane curve described by parametric equations? Give an example with your description.

73. What is the significance of arrows along a plane curve?

74. What does it mean to eliminate the parameter? What useful information can be obtained by doing this?

75. Explain how the rectangular equation $y = 5x$ can have infinitely many sets of parametric equations.

76. Discuss how the parametric equations for the path of a projectile (see Exercises 69–70) and the ability to obtain plane curves with a graphing utility can be used by a baseball coach to analyze performances of team players.

Technology Exercises

77. Use a graphing utility in a parametric mode to verify any five of your hand-drawn graphs in Exercises 9–40.

In Exercises 78–82, use a graphing utility to obtain the plane curve represented by the given parametric equations.

78. Cycloid: $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$;
$[0, 60, 5] \times [0, 8, 1], 0 \le t < 6\pi$

79. Cycloid: $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$;
$[0, 60, 5] \times [0, 8, 1], 0 \le t < 6\pi$

80. Witch of Agnesi: $x = 2\cot t$, $y = 2\sin^2 t$;
$[-6, 6, 1] \times [-4, 4, 1], 0 \le t < 2\pi$

81. Hypocycloid: $x = 4\cos^3 t$, $y = 4\sin^3 t$;
$[-5, 5, 1] \times [-5, 5, 1], 0 \le t < 2\pi$

82. Lissajous Curve: $x = 2\cos t$, $y = \sin 2t$;
$[-3, 3, 1] \times [-2, 2, 1], 0 \le t < 2\pi$

Use the equations for the path of a projectile given prior to Exercises 69–70 to solve Exercises 83–85.

In Exercises 83–84, use a graphing utility to obtain the path of a projectile launched from the ground ($h = 0$) at the specified values of θ and v_0. In each exercise, use the graph to determine the maximum height and the time at which the projectile reaches its maximum height. Also use the graph to determine the range of the projectile and the time it hits the ground. Round all answers to the nearest tenth.

83. $\theta = 55°$, $v_0 = 200$ feet per second

84. $\theta = 35°$, $v_0 = 300$ feet per second

85. A baseball player throws a ball with an initial velocity of 140 feet per second at an angle of 22° to the horizontal. The ball leaves the player's hand at a height of 5 feet.

a. Write the parametric equations that describe the ball's position as a function of time.

b. Use a graphing utility to obtain the path of the baseball.

c. Find the ball's maximum height and the time at which it reaches this height. Round all answers to the nearest tenth.

d. How long is the ball in the air?

e. How far does the ball travel?

Critical Thinking Exercises

Make Sense? *In Exercises 86–89, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

86. Parametric equations allow me to use functions to describe curves that are not graphs of functions.

87. Parametric equations let me think of a curve as a path traced out by a moving point.

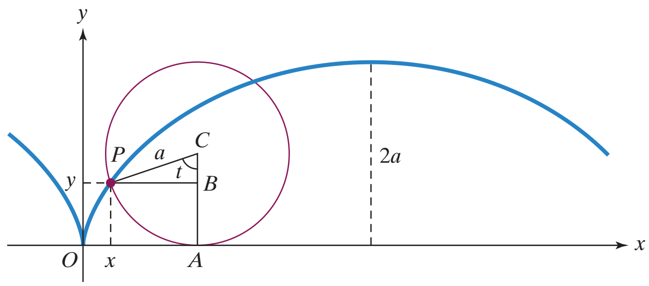
88. I represented $y = x^2 - 9$ with the parametric equations $x = t^2$ and $y = t^4 - 9$.

89. I found alternate pairs of parametric equations for the same rectangular equation.

90. Eliminate the parameter: $x = \cos^3 t$ and $y = \sin^3 t$.

91. The plane curve described by the parametric equations $x = 3\cos t$ and $y = 3\sin t$, $0 \le t < 2\pi$, has a counterclockwise orientation. Alter one or both parametric equations so that you obtain the same plane curve with the opposite orientation.

92. The figure shows a circle of radius a rolling along a horizontal line. Point P traces out a cycloid. Angle t, in radians, is the angle through which the circle has rolled. C is the center of the circle.



Use the suggestions in parts (a) and (b) to prove that the parametric equations of the cycloid are $x = a(t - \sin t)$ and $y = a(1 - \cos t)$.

a. Derive the parametric equation for x using the figure and
$$x = OA - xA.$$

b. Derive the parametric equation for y using the figure and
$$y = AC - BC.$$

Preview Exercises

Exercises 93–95 will help you prepare for the material covered in the next section.

93. Rewrite $r = \dfrac{4}{2 + \cos\theta}$ by dividing the numerator and the denominator by 2.

94. Complete the table of coordinates at the top of the next page. Where necessary, round to two decimal places. Then plot the resulting points, (r, θ), using a polar coordinate system.

θ	0	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = \frac{4}{2 + \cos\theta}$						

95. a. Showing all steps, rewrite $r = \frac{1}{3 - 3\cos\theta}$ as $9r^2 = (1 + 3r\cos\theta)^2$.

b. Express $9r^2 = (1 + 3r\cos\theta)^2$ in rectangular coordinates. Which conic section is represented by the rectangular equation?

Section 9.6 Conic Sections in Polar Coordinates

Objectives

1. Define conics in terms of a focus and a directrix.
2. Graph the polar equations of conics.

John Glenn made the first U.S.-manned flight around Earth on *Friendship 7*.

On the morning of February 20, 1962, millions of Americans collectively held their breath as the world's newest pioneer swept across the threshold of one of our last frontiers. Roughly one hundred miles above Earth, astronaut John Glenn sat comfortably in the weightless environment of a $9\frac{1}{2}$-by-6-foot space capsule that offered the leg room of a Volkswagen "Beetle" and the aesthetics of a garbage can. Glenn became the first American to orbit Earth in a three-orbit mission that lasted slightly under 5 hours.

In this section's exercise set, you will see how John Glenn's historic orbit can be described using conic sections in polar coordinates. To obtain this model, we begin with a definition that permits a unified approach to the conic sections.

1 Define conics in terms of a focus and a directrix.

The Focus-Directrix Definitions of the Conic Sections

The definition of a parabola is given in terms of a fixed point, the focus, and a fixed line, the directrix. By contrast, the definitions of an ellipse and a hyperbola are given in terms of two fixed points, the foci. It is possible to define each of these conic sections in terms of a point and a line. **Figure 9.56** shows a conic section in the polar coordinate system. The fixed point, the focus, is at the pole. The fixed line, the directrix, is perpendicular to the polar axis.

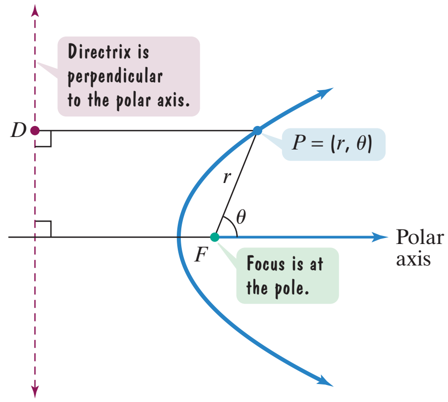


Figure 9.56 A conic in the polar coordinate system

Focus-Directrix Definitions of the Conic Sections

Let F be a fixed point, the focus, and let D be a fixed line, the directrix, in a plane (**Figure 9.56**). A **conic section**, or **conic**, is the set of all points P in the plane such that

$$\frac{PF}{PD} = e,$$

where e is a fixed positive number, called the **eccentricity**.

If $e = 1$, the conic is a parabola.

If $e < 1$, the conic is an ellipse.

If $e > 1$, the conic is a hyperbola.