129. Explain how to convert from scientific to decimal notation and give an example.

130. Explain how to convert from decimal to scientific notation and give an example.

Critical Thinking Exercises

Make Sense? *In Exercises 131–134, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

131. There are many exponential expressions that are equal to $36x^{12}$, such as $(6x^6)^2$, $(6x^3)(6x^9)$, $36(x^3)^9$, and $6^2(x^2)^6$.

132. If 5^{-2} is raised to the third power, the result is a number between 0 and 1.

133. The population of Colorado is approximately 4.6×10^{12}.

134. I just finished reading a book that contained approximately 1.04×10^5 words.

In Exercises 135–142, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

135. $4^{-2} < 4^{-3}$

136. $5^{-2} > 2^{-5}$

137. $(-2)^4 = 2^{-4}$

138. $5^2 \cdot 5^{-2} > 2^5 \cdot 2^{-5}$

139. $534.7 = 5.347 \times 10^3$

140. $\dfrac{8 \times 10^{30}}{4 \times 10^{-5}} = 2 \times 10^{25}$

141. $(7 \times 10^5) + (2 \times 10^{-3}) = 9 \times 10^2$

142. $(4 \times 10^3) + (3 \times 10^2) = 4.3 \times 10^3$

143. The mad Dr. Frankenstein has gathered enough bits and pieces (so to speak) for $2^{-1} + 2^{-2}$ of his creature-to-be. Write a fraction that represents the amount of his creature that must still be obtained.

144. If $b^A = MN$, $b^C = M$, and $b^D = N$, what is the relationship among A, C, and D?

145. Our hearts beat approximately 70 times per minute. Express in scientific notation how many times the heart beats over a lifetime of 80 years. Round the decimal factor in your scientific notation answer to two decimal places.

Group Exercise

146. Putting Numbers into Perspective. A large number can be put into perspective by comparing it with another number. For example, we put the \$9.2 trillion national debt (Example 6) and the \$2.52 trillion the government collected in taxes (Exercise 115) by comparing these numbers to the number of U.S. citizens.

For this project, each group member should consult an almanac, a newspaper, or the World Wide Web to find a number greater than one million. Explain to other members of the group the context in which the large number is used. Express the number in scientific notation. Then put the number into perspective by comparing it with another number.

Preview Exercises

Exercises 147–149 will help you prepare for the material covered in the next section.

147. **a.** Find $\sqrt{16} \cdot \sqrt{4}$.
b. Find $\sqrt{16 \cdot 4}$.
c. Based on your answers to parts (a) and (b), what can you conclude?

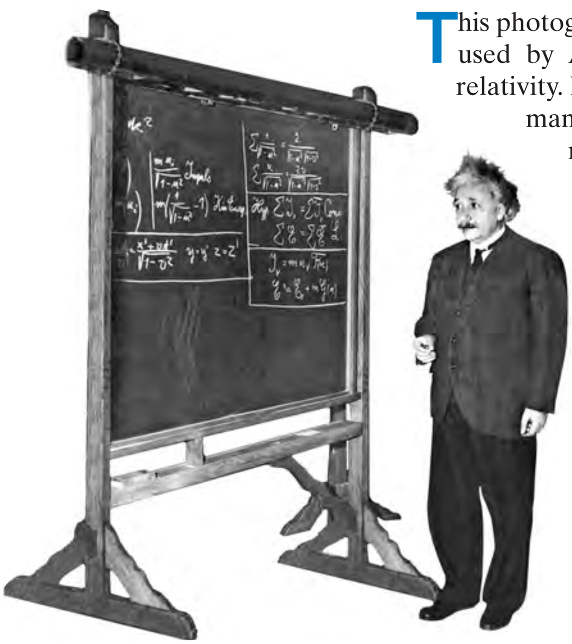
148. **a.** Use a calculator to approximate $\sqrt{300}$ to two decimal places.
b. Use a calculator to approximate $10\sqrt{3}$ to two decimal places.
c. Based on your answers to parts (a) and (b), what can you conclude?

149. **a.** Simplify: $21x + 10x$.
b. Simplify: $21\sqrt{2} + 10\sqrt{2}$.

Section P.3 Radicals and Rational Exponents

Objectives

1. Evaluate square roots.
2. Simplify expressions of the form $\sqrt{a^2}$.
3. Use the product rule to simplify square roots.
4. Use the quotient rule to simplify square roots.
5. Add and subtract square roots.
6. Rationalize denominators.
7. Evaluate and perform operations with higher roots.
8. Understand and use rational exponents.



This photograph shows mathematical models used by Albert Einstein at a lecture on relativity. Notice the radicals that appear in many of the formulas. Among these models, there is one describing how an astronaut in a moving spaceship ages more slowly than friends who remain on Earth. No description of your world can be complete without roots and radicals. In this section, in addition to reviewing the basics of radical expressions and the use of rational exponents to indicate radicals, you will see how radicals model time dilation for a futuristic high-speed trip to a nearby star.

1 Evaluate square roots.

Square Roots

From our earlier work with exponents, we are aware that the square of both 5 and -5 is 25:

$$5^2 = 25 \quad \text{and} \quad (-5)^2 = 25.$$

The reverse operation of squaring a number is finding the *square root* of the number. For example,

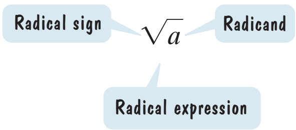
- One square root of 25 is 5 because $5^2 = 25$.
- Another square root of 25 is -5 because $(-5)^2 = 25$.

In general, **if $b^2 = a$, then b is a square root of a.**

The symbol $\sqrt{}$ is used to denote the *positive* or *principal square root* of a number. For example,

- $\sqrt{25} = 5$ because $5^2 = 25$ and 5 is positive.
- $\sqrt{100} = 10$ because $10^2 = 100$ and 10 is positive.

The symbol $\sqrt{}$ that we use to denote the principal square root is called a **radical sign**. The number under the radical sign is called the **radicand**. Together we refer to the radical sign and its radicand as a **radical expression**.



Definition of the Principal Square Root

If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$, denoted by $b = \sqrt{a}$, is the **principal square root** of a.

The symbol $-\sqrt{}$ is used to denote the negative square root of a number. For example,

- $-\sqrt{25} = -5$ because $(-5)^2 = 25$ and -5 is negative.
- $-\sqrt{100} = -10$ because $(-10)^2 = 100$ and -10 is negative.

EXAMPLE 1 Evaluating Square Roots

Evaluate:

a. $\sqrt{64}$ **b.** $-\sqrt{49}$ **c.** $\sqrt{\frac{1}{4}}$ **d.** $\sqrt{9+16}$ **e.** $\sqrt{9} + \sqrt{16}$.

Solution

a. $\sqrt{64} = 8$ The principal square root of 64 is 8. Check: $8^2 = 64$.

b. $-\sqrt{49} = -7$ The negative square root of 49 is -7. Check: $(-7)^2 = 49$.

c. $\sqrt{\frac{1}{4}} = \frac{1}{2}$ The principal square root of $\frac{1}{4}$ is $\frac{1}{2}$. Check: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

d. $\sqrt{9+16} = \sqrt{25}$ First simplify the expression under the radical sign.

$= 5$ Then take the principal square root of 25, which is 5.

e. $\sqrt{9} + \sqrt{16} = 3 + 4$ $\sqrt{9} = 3$ because $3^2 = 9$. $\sqrt{16} = 4$ because $4^2 = 16$.

$= 7$

Study Tip

In Example 1, parts (d) and (e), observe that $\sqrt{9+16}$ is not equal to $\sqrt{9} + \sqrt{16}$. In general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

and

$$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}.$$

Check Point 1 Evaluate:

a. $\sqrt{81}$ **b.** $-\sqrt{9}$ **c.** $\sqrt{\frac{1}{25}}$

d. $\sqrt{36 + 64}$ **e.** $\sqrt{36} + \sqrt{64}$.

A number that is the square of a rational number is called a **perfect square**. All the radicands in Example 1 and Check Point 1 are perfect squares.

- 64 is a perfect square because $64 = 8^2$. Thus, $\sqrt{64} = 8$.
- $\frac{1}{4}$ is a perfect square because $\frac{1}{4} = \left(\frac{1}{2}\right)^2$. Thus, $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Let's see what happens to the radical expression $\sqrt{x}$ if x is a negative number. Is the square root of a negative number a real number? For example, consider $\sqrt{-25}$. Is there a real number whose square is -25? No. Thus, $\sqrt{-25}$ is not a real number. In general, **a square root of a negative number is not a real number**.

If a number a is nonnegative $(a \geq 0)$, then $(\sqrt{a})^2 = a$. For example,

$$(\sqrt{2})^2 = 2, \quad (\sqrt{3})^2 = 3, \quad (\sqrt{4})^2 = 4, \quad \text{and} \quad (\sqrt{5})^2 = 5.$$

2 Simplify expressions of the form $\sqrt{a^2}$.

Simplifying Expressions of the Form $\sqrt{a^2}$

You may think that $\sqrt{a^2} = a$. However, this is not necessarily true. Consider the following examples:

$$\sqrt{4^2} = \sqrt{16} = 4$$
$$\sqrt{(-4)^2} = \sqrt{16} = 4.$$

The result is not -4, but rather the absolute value of -4, or 4.

Here is a rule for simplifying expressions of the form $\sqrt{a^2}$:

Simplifying $\sqrt{a^2}$

For any real number a,

$$\sqrt{a^2} = |a|.$$

In words, the principal square root of a^2 is the absolute value of a.

For example, $\sqrt{6^2} = |6| = 6$ and $\sqrt{(-6)^2} = |-6| = 6$.

3 Use the product rule to simplify square roots.

The Product Rule for Square Roots

A rule for multiplying square roots can be generalized by comparing $\sqrt{25} \cdot \sqrt{4}$ and $\sqrt{25 \cdot 4}$. Notice that

$$\sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10 \quad \text{and} \quad \sqrt{25 \cdot 4} = \sqrt{100} = 10.$$

Because we obtain 10 in both situations, the original radical expressions must be equal. That is,

$$\sqrt{25} \cdot \sqrt{4} = \sqrt{25 \cdot 4}.$$

This result is a special case of the **product rule for square roots** that can be generalized as follows:

The Product Rule for Square Roots

If a and b represent nonnegative real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

The square root of a product is the product of the square roots.

The product of two square roots is the square root of the product of the radicands.

A square root is **simplified** when its radicand has no factors other than 1 that are perfect squares. For example, $\sqrt{500}$ is not simplified because it can be expressed as $\sqrt{100 \cdot 5}$ and 100 is a perfect square. Example 2 shows how the product rule is used to remove from the square root any perfect squares that occur as factors.

EXAMPLE 2 Using the Product Rule to Simplify Square Roots

Simplify: **a.** $\sqrt{500}$ **b.** $\sqrt{6x} \cdot \sqrt{3x}$.

Solution

a. $\sqrt{500} = \sqrt{100 \cdot 5}$ — Factor 500; 100 is the greatest perfect square factor.

$= \sqrt{100}\sqrt{5}$ — Use the product rule: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

$= 10\sqrt{5}$ — Write $\sqrt{100}$ as 10. We read $10\sqrt{5}$ as "ten times the square root of 5."

b. We can simplify $\sqrt{6x} \cdot \sqrt{3x}$ using the product rule only if $6x$ and $3x$ represent nonnegative real numbers. Thus, $x \geq 0$.

$\sqrt{6x} \cdot \sqrt{3x} = \sqrt{6x \cdot 3x}$ — Use the product rule: $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

$= \sqrt{18x^2}$ — Multiply in the radicand.

$= \sqrt{9x^2 \cdot 2}$ — Factor 18; 9 is the greatest perfect square factor.

$= \sqrt{9x^2}\sqrt{2}$ — Use the product rule: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

$= \sqrt{9}\sqrt{x^2}\sqrt{2}$ — Use the product rule to write $\sqrt{9x^2}$ as the product of two square roots.

$= 3x\sqrt{2}$ — $\sqrt{x^2} = |x| = x$ because $x \geq 0$.

Study Tip

When simplifying square roots, always look for the *greatest* perfect square factor possible. The following factorization will lead to further simplification:

$$\sqrt{500} = \sqrt{25 \cdot 20} = \sqrt{25}\sqrt{20} = 5\sqrt{20}.$$

25 is a perfect square factor of 500, but not the greatest perfect square factor.

Because 20 contains a perfect square factor, 4, the simplification is not complete.

$$5\sqrt{20} = 5\sqrt{4 \cdot 5} = 5\sqrt{4}\sqrt{5} = 5 \cdot 2\sqrt{5} = 10\sqrt{5}$$

Although the result checks with our simplification using $\sqrt{500} = \sqrt{100 \cdot 5}$, more work is required when the greatest perfect square factor is not used.

Check Point 2 Simplify:

a. $\sqrt{75}$ **b.** $\sqrt{5x} \cdot \sqrt{10x}$.

4 Use the quotient rule to simplify square roots.

The Quotient Rule for Square Roots

Another property for square roots involves division.

The Quotient Rule for Square Roots

If a and b represent nonnegative real numbers and $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

The square root of a quotient is the quotient of the square roots.

The quotient of two square roots is the square root of the quotient of the radicands.

EXAMPLE 3 Using the Quotient Rule to Simplify Square Roots

Simplify: **a.** $\sqrt{\dfrac{100}{9}}$ **b.** $\dfrac{\sqrt{48x^3}}{\sqrt{6x}}$.

Solution

a. $\sqrt{\dfrac{100}{9}} = \dfrac{\sqrt{100}}{\sqrt{9}} = \dfrac{10}{3}$

b. We can simplify the quotient of $\sqrt{48x^3}$ and $\sqrt{6x}$ using the quotient rule only if $48x^3$ and $6x$ represent nonnegative real numbers and $6x \neq 0$. Thus, $x > 0$.

$$\frac{\sqrt{48x^3}}{\sqrt{6x}} = \sqrt{\frac{48x^3}{6x}} = \sqrt{8x^2} = \sqrt{4x^2}\sqrt{2} = \sqrt{4}\sqrt{x^2}\sqrt{2} = 2x\sqrt{2}$$

$\sqrt{x^2} = |x| = x$ because $x > 0$.

Check Point 3 Simplify:

a. $\sqrt{\dfrac{25}{16}}$ **b.** $\dfrac{\sqrt{150x^3}}{\sqrt{2x}}$.

5 Add and subtract square roots.

Adding and Subtracting Square Roots

Two or more square roots can be combined using the distributive property provided that they have the same radicand. Such radicals are called **like radicals**. For example,

$$7\sqrt{11} + 6\sqrt{11} = (7 + 6)\sqrt{11} = 13\sqrt{11}.$$

7 square roots of 11 plus 6 square roots of 11 result in 13 square roots of 11.

EXAMPLE 4 Adding and Subtracting Like Radicals

Add or subtract as indicated:

a. $7\sqrt{2} + 5\sqrt{2}$ **b.** $\sqrt{5x} - 7\sqrt{5x}$.

Solution

a. $7\sqrt{2} + 5\sqrt{2} = (7 + 5)\sqrt{2}$ Apply the distributive property.

$= 12\sqrt{2}$ Simplify.

b. $\sqrt{5x} - 7\sqrt{5x} = 1\sqrt{5x} - 7\sqrt{5x}$ Write $\sqrt{5x}$ as $1\sqrt{5x}$.

$= (1 - 7)\sqrt{5x}$ Apply the distributive property.

$= -6\sqrt{5x}$ Simplify.

Check Point 4 Add or subtract as indicated:

a. $8\sqrt{13} + 9\sqrt{13}$ **b.** $\sqrt{17x} - 20\sqrt{17x}$.

In some cases, radicals can be combined once they have been simplified. For example, to add $\sqrt{2}$ and $\sqrt{8}$, we can write $\sqrt{8}$ as $\sqrt{4 \cdot 2}$ because 4 is a perfect square factor of 8.

$$\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4 \cdot 2} = 1\sqrt{2} + 2\sqrt{2} = (1 + 2)\sqrt{2} = 3\sqrt{2}$$

EXAMPLE 5 Combining Radicals That First Require Simplification

Add or subtract as indicated:

a. $7\sqrt{3} + \sqrt{12}$ **b.** $4\sqrt{50x} - 6\sqrt{32x}$.

Solution

a. $7\sqrt{3} + \sqrt{12}$

$= 7\sqrt{3} + \sqrt{4 \cdot 3}$ Split 12 into two factors such that one is a perfect square.

$= 7\sqrt{3} + 2\sqrt{3}$ $\sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

$= (7 + 2)\sqrt{3}$ Apply the distributive property. You will find that this step is usually done mentally.

$= 9\sqrt{3}$ Simplify.

b. $4\sqrt{50x} - 6\sqrt{32x}$

$= 4\sqrt{25 \cdot 2x} - 6\sqrt{16 \cdot 2x}$ 25 is the greatest perfect square factor of 50x and 16 is the greatest perfect square factor of 32x.

$= 4 \cdot 5\sqrt{2x} - 6 \cdot 4\sqrt{2x}$ $\sqrt{25 \cdot 2x} = \sqrt{25}\sqrt{2x} = 5\sqrt{2x}$ and $\sqrt{16 \cdot 2x} = \sqrt{16}\sqrt{2x} = 4\sqrt{2x}$.

$= 20\sqrt{2x} - 24\sqrt{2x}$ Multiply: $4 \cdot 5 = 20$ and $6 \cdot 4 = 24$.

$= (20 - 24)\sqrt{2x}$ Apply the distributive property.

$= -4\sqrt{2x}$ Simplify.

Check Point 5 Add or subtract as indicated:

a. $5\sqrt{27} + \sqrt{12}$ **b.** $6\sqrt{18x} - 4\sqrt{8x}$.

6 Rationalize denominators.

Rationalizing Denominators

The calculator screen in **Figure P.11** shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$. The two approximations are the same. This is not a coincidence:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \boxed{\frac{\sqrt{3}}{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}.$$

Any number divided by itself is 1. Multiplication by 1 does not change the value of $\frac{1}{\sqrt{3}}$.

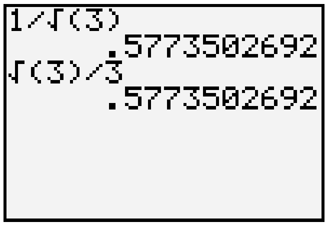


Figure P.11 The calculator screen shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$.

This process involves rewriting a radical expression as an equivalent expression in which the denominator no longer contains any radicals. The process is called **rationalizing the denominator**. If the denominator consists of the square root of a natural number that is not a perfect square, **multiply the numerator and the denominator by the smallest number that produces the square root of a perfect square in the denominator**.

EXAMPLE 6 Rationalizing Denominators

Rationalize the denominator:

a. $\frac{15}{\sqrt{6}}$ **b.** $\frac{12}{\sqrt{8}}$.

Study Tip

Rationalizing a numerical denominator makes that denominator a rational number.

Solution

a. If we multiply the numerator and the denominator of $\frac{15}{\sqrt{6}}$ by $\sqrt{6}$, the denominator becomes $\sqrt{6}\cdot\sqrt{6} = \sqrt{36} = 6$. Therefore, we multiply by 1, choosing $\frac{\sqrt{6}}{\sqrt{6}}$ for 1.

$$\frac{15}{\sqrt{6}} = \underbrace{\frac{15}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}}_{\text{Multiply by 1.}} = \frac{15\sqrt{6}}{\sqrt{36}} = \frac{15\sqrt{6}}{6} = \underbrace{\frac{5\sqrt{6}}{2}}_{\text{Simplify: } \frac{15}{6} = \frac{15 \div 3}{6 \div 3} = \frac{5}{2}.}$$

b. The *smallest* number that will produce the square root of a perfect square in the denominator of $\frac{12}{\sqrt{8}}$ is $\sqrt{2}$, because $\sqrt{8}\cdot\sqrt{2} = \sqrt{16} = 4$. We multiply by 1, choosing $\frac{\sqrt{2}}{\sqrt{2}}$ for 1.

$$\frac{12}{\sqrt{8}} = \frac{12}{\sqrt{8}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{\sqrt{16}} = \frac{12\sqrt{2}}{4} = 3\sqrt{2}$$

Check Point 6 Rationalize the denominator:

a. $\frac{5}{\sqrt{3}}$ **b.** $\frac{6}{\sqrt{12}}$.

Radical expressions that involve the sum and difference of the same two terms are called **conjugates**. Thus,

$$\sqrt{a} + \sqrt{b} \quad \text{and} \quad \sqrt{a} - \sqrt{b}$$

are conjugates. Conjugates are used to rationalize denominators because the product of such a pair contains no radicals:

$$\begin{aligned}
&\underbrace{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})}_{\text{Multiply each term of } \sqrt{a}-\sqrt{b} \text{ by each term of } \sqrt{a}+\sqrt{b}.}\\
&= \underbrace{\sqrt{a}(\sqrt{a} - \sqrt{b})}_{\text{Distribute } \sqrt{a} \text{ over } \sqrt{a}-\sqrt{b}.} + \underbrace{\sqrt{b}(\sqrt{a} - \sqrt{b})}_{\text{Distribute } \sqrt{b} \text{ over } \sqrt{a}-\sqrt{b}.}\\
&= \sqrt{a}\cdot\sqrt{a} - \sqrt{a}\cdot\sqrt{b} + \sqrt{b}\cdot\sqrt{a} - \sqrt{b}\cdot\sqrt{b}\\
&= (\sqrt{a})^2 \underbrace{- \sqrt{ab} + \sqrt{ab}}_{-\sqrt{ab}+\sqrt{ab}=0} - (\sqrt{b})^2\\
&= (\sqrt{a})^2 - (\sqrt{b})^2\\
&= a - b.
\end{aligned}$$

Multiplying Conjugates

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$

How can we rationalize a denominator if the denominator contains two terms with one or more square roots? **Multiply the numerator and the**

denominator by the conjugate of the denominator. Here are three examples of such expressions:

- $\dfrac{7}{5+\sqrt{3}}$ The conjugate of the denominator is $5-\sqrt{3}$.
- $\dfrac{8}{3\sqrt{2}-4}$ The conjugate of the denominator is $3\sqrt{2}+4$.
- $\dfrac{h}{\sqrt{x+h}-\sqrt{x}}$ The conjugate of the denominator is $\sqrt{x+h}+\sqrt{x}$.

The product of the denominator and its conjugate is found using the formula

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2 = a-b.$$

The simplified product will not contain a radical.

EXAMPLE 7 Rationalizing a Denominator Containing Two Terms

Rationalize the denominator: $\dfrac{7}{5+\sqrt{3}}$.

Solution The conjugate of the denominator is $5-\sqrt{3}$. If we multiply the numerator and denominator by $5-\sqrt{3}$, the simplified denominator will not contain a radical. Therefore, we multiply by 1, choosing $\dfrac{5-\sqrt{3}}{5-\sqrt{3}}$ for 1.

$$\frac{7}{5+\sqrt{3}} = \frac{7}{5+\sqrt{3}}\cdot\frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{7(5-\sqrt{3})}{5^2-(\sqrt{3})^2} = \frac{7(5-\sqrt{3})}{25-3}$$

Multiply by 1.

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2$

$$= \frac{7(5-\sqrt{3})}{22} \quad\text{or}\quad \frac{35-7\sqrt{3}}{22}$$

In either form of the answer, there is no radical in the denominator.

Check Point 7 Rationalize the denominator: $\dfrac{8}{4+\sqrt{5}}$.

7 Evaluate and perform operations with higher roots.

Other Kinds of Roots

We define the **principal *n*th root** of a real number a, symbolized by $\sqrt[n]{a}$, as follows:

Definition of the Principal *n*th Root of a Real Number

$$\sqrt[n]{a} = b \text{ means that } b^n = a.$$

If n, the **index**, is even, then a is nonnegative $(a \ge 0)$ and b is also nonnegative $(b \ge 0)$. If n is odd, a and b can be any real numbers.

For example,

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64 \quad\text{and}\quad \sqrt[5]{-32} = -2 \text{ because } (-2)^5 = -32.$$

The same vocabulary that we learned for square roots applies to nth roots. The symbol $\sqrt[n]{}$ is called a **radical** and the expression under the radical is called the **radicand**.

Study Tip

Some higher even and odd roots occur so frequently that you might want to memorize them.

Cube Roots	
$\sqrt[3]{1} = 1$	$\sqrt[3]{125} = 5$
$\sqrt[3]{8} = 2$	$\sqrt[3]{216} = 6$
$\sqrt[3]{27} = 3$	$\sqrt[3]{1000} = 10$
$\sqrt[3]{64} = 4$	

Fourth Roots
$\sqrt[4]{1} = 1$
$\sqrt[4]{16} = 2$
$\sqrt[4]{81} = 3$
$\sqrt[4]{256} = 4$
$\sqrt[4]{625} = 5$

Fifth Roots
$\sqrt[5]{1} = 1$
$\sqrt[5]{32} = 2$
$\sqrt[5]{243} = 3$

A number that is the nth power of a rational number is called a **perfect *n*th power**. For example, 8 is a perfect third power, or perfect cube, because $8 = 2^3$. Thus, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. In general, one of the following rules can be used to find the nth root of a perfect nth power:

Finding *n*th Roots of Perfect *n*th Powers

If n is odd, $\sqrt[n]{a^n} = a$.
If n is even, $\sqrt[n]{a^n} = |a|$.

For example,

$$\sqrt[3]{(-2)^3} = -2 \qquad \text{and} \qquad \sqrt[4]{(-2)^4} = |-2| = 2.$$

Absolute value is not needed with odd roots, but is necessary with even roots.

The Product and Quotient Rules for Other Roots

The product and quotient rules apply to cube roots, fourth roots, and all higher roots.

The Product and Quotient Rules for *n*th Roots

For all real numbers a and b, where the indicated roots represent real numbers,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad \text{and} \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

The nth root of a product is the product of the nth roots.

The product of two nth roots is the nth root of the product of the radicands.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},\ b \neq 0 \qquad \text{and} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq 0.$$

The nth root of a quotient is the quotient of the nth roots.

The quotient of two nth roots is the nth root of the quotient of the radicands.

EXAMPLE 8 Simplifying, Multiplying, and Dividing Higher Roots

Simplify: **a.** $\sqrt[3]{24}$ **b.** $\sqrt[4]{8} \cdot \sqrt[4]{4}$ **c.** $\sqrt[4]{\frac{81}{16}}$.

Solution

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}$ Find the greatest perfect cube that is a factor of 24; $2^3 = 8$, so 8 is a perfect cube and is the greatest perfect cube factor of 24.

$= \sqrt[3]{8} \cdot \sqrt[3]{3}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[3]{3}$ $\sqrt[3]{8} = 2$

b. $\sqrt[4]{8} \cdot \sqrt[4]{4} = \sqrt[4]{8 \cdot 4}$ $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

$= \sqrt[4]{32}$ Find the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16 \cdot 2}$ $2^4 = 16$, so 16 is a perfect fourth power and is the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16} \cdot \sqrt[4]{2}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[4]{2}$ $\sqrt[4]{16} = 2$

c. $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}}$ $\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$= \frac{3}{2}$ $\qquad \sqrt[4]{81} = 3$ because $3^4 = 81$ and $\sqrt[4]{16} = 2$ because $2^4 = 16$.

Check Point 8 Simplify:

a. $\sqrt[3]{40}$ **b.** $\sqrt[5]{8} \cdot \sqrt[5]{8}$ **c.** $\sqrt[3]{\frac{125}{27}}$.

We have seen that adding and subtracting square roots often involves simplifying terms. The same idea applies to adding and subtracting *n*th roots.

EXAMPLE 9 Combining Cube Roots

Subtract: $5\sqrt[3]{16} - 11\sqrt[3]{2}$.

Solution

$5\sqrt[3]{16} - 11\sqrt[3]{2}$

$= 5\sqrt[3]{8 \cdot 2} - 11\sqrt[3]{2}$ Factor 16; 8 is the greatest perfect cube factor: $2^3 = 8$ and $\sqrt[3]{8} = 2$.

$= 5 \cdot 2\sqrt[3]{2} - 11\sqrt[3]{2}$ $\sqrt[3]{8 \cdot 2} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$

$= 10\sqrt[3]{2} - 11\sqrt[3]{2}$ Multiply: $5 \cdot 2 = 10$.

$= (10 - 11)\sqrt[3]{2}$ Apply the distributive property.

$= -1\sqrt[3]{2}$ or $-\sqrt[3]{2}$ Simplify.

Check Point 9 Subtract: $3\sqrt[3]{81} - 4\sqrt[3]{3}$.

8 Understand and use rational exponents.

Rational Exponents

We define rational exponents so that their properties are the same as the properties for integer exponents. For example, we know that exponents are multiplied when an exponential expression is raised to a power. For this to be true,

$$\left(7^{\frac{1}{2}}\right)^2 = 7^{\frac{1}{2}\cdot 2} = 7^1 = 7.$$

We also know that

$$\left(\sqrt{7}\right)^2 = \sqrt{7} \cdot \sqrt{7} = \sqrt{49} = 7.$$

Can you see that the square of both $7^{\frac{1}{2}}$ and $\sqrt{7}$ is 7? It is reasonable to conclude that

$$7^{\frac{1}{2}} \quad \text{means} \quad \sqrt{7}.$$

We can generalize the fact that $7^{\frac{1}{2}}$ means $\sqrt{7}$ with the following definition:

The Definition of $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ represents a real number, where $n \geq 2$ is an integer, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

The denominator of the rational exponent is the radical's index.

Furthermore,

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{a}}, \quad a \neq 0.$$

EXAMPLE 10 Using the Definition of $a^{\frac{1}{n}}$

Simplify:

a. $64^{\frac{1}{2}}$ **b.** $125^{\frac{1}{3}}$ **c.** $-16^{\frac{1}{4}}$ **d.** $(-27)^{\frac{1}{3}}$ **e.** $64^{-\frac{1}{3}}$.

Solution

a. $64^{\frac{1}{2}} = \sqrt{64} = 8$

b. $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

The denominator is the index.

c. $-16^{\frac{1}{4}} = -(\sqrt[4]{16}) = -2$

The base is 16 and the negative sign is not affected by the exponent.

d. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Parentheses show that the base is −27 and that the negative sign is affected by the exponent.

e. $64^{-\frac{1}{3}} = \dfrac{1}{64^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{64}} = \dfrac{1}{4}$

Check Point 10 Simplify:

a. $25^{\frac{1}{2}}$ **b.** $8^{\frac{1}{3}}$ **c.** $-81^{\frac{1}{4}}$ **d.** $(-8)^{\frac{1}{3}}$ **e.** $27^{-\frac{1}{3}}$.

In Example 10 and Check Point 10, each rational exponent had a numerator of 1. If the numerator is some other integer, we still want to multiply exponents when raising a power to a power. For this reason,

$$a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 \quad \text{and} \quad a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}.$$

This means $(\sqrt[3]{a})^2$. This means $\sqrt[3]{a^2}$.

Thus,

$$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2 = \sqrt[3]{a^2}.$$

Do you see that the denominator, 3, of the rational exponent is the same as the index of the radical? The numerator, 2, of the rational exponent serves as an exponent in each of the two radical forms. We generalize these ideas with the following definition:

The Definition of $a^{\frac{m}{n}}$

If $\sqrt[n]{a}$ represents a real number and $\dfrac{m}{n}$ is a positive rational number, $n \geq 2$, then

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m.$$

Also,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Furthermore, if $a^{-\frac{m}{n}}$ is a nonzero real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$

The first form of the definition of $a^{\frac{m}{n}}$, shown again below, involves taking the root first. This form is often preferable because smaller numbers are involved. Notice that the rational exponent consists of two parts, indicated by the following voice balloons:

The numerator is the exponent.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

The denominator is the radical's index.

EXAMPLE 11 Using the Definition of $a^{\frac{m}{n}}$

Simplify: **a.** $27^{\frac{2}{3}}$ **b.** $9^{\frac{3}{2}}$ **c.** $81^{-\frac{3}{4}}$.

Solution

a. $27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

b. $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$

c. $81^{-\frac{3}{4}} = \dfrac{1}{81^{\frac{3}{4}}} = \dfrac{1}{\left(\sqrt[4]{81}\right)^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

Technology

Here are the calculator keystroke sequences for $81^{-\frac{3}{4}}$:

Many Scientific Calculators

81 $\boxed{y^x}$ $\boxed{(}$ 3 $\boxed{+/-}$ $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{=}$

Many Graphing Calculators

81 $\boxed{\wedge}$ $\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$.

Check Point 11 Simplify:

a. $27^{\frac{4}{3}}$ **b.** $4^{\frac{3}{2}}$ **c.** $32^{-\frac{2}{5}}$.

Properties of exponents can be applied to expressions containing rational exponents.

EXAMPLE 12 Simplifying Expressions with Rational Exponents

Simplify using properties of exponents:

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right)$ **b.** $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}}$.

Solution

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right) = 5 \cdot 7x^{\frac{1}{2}} \cdot x^{\frac{3}{4}}$ Group numerical factors and group variable factors with the same base.

$= 35x^{\frac{1}{2}+\frac{3}{4}}$ When multiplying expressions with the same base, add the exponents.

$= 35x^{\frac{5}{4}}$ $\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$

b. $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}} = \left(\dfrac{32}{16}\right)\left(\dfrac{x^{\frac{5}{3}}}{x^{\frac{3}{4}}}\right)$ Group numerical factors and group variable factors with the same base.

$= 2x^{\frac{5}{3}-\frac{3}{4}}$ When dividing expressions with the same base, subtract the exponents.

$= 2x^{\frac{11}{12}}$ $\frac{5}{3} - \frac{3}{4} = \frac{20}{12} - \frac{9}{12} = \frac{11}{12}$

Check Point 12 Simplify using properties of exponents:

a. $\left(2x^{\frac{4}{3}}\right)\left(5x^{\frac{8}{3}}\right)$ **b.** $\dfrac{20x^4}{5x^{\frac{3}{2}}}$.

Rational exponents are sometimes useful for simplifying radicals by reducing the index.

EXAMPLE 13 Reducing the Index of a Radical

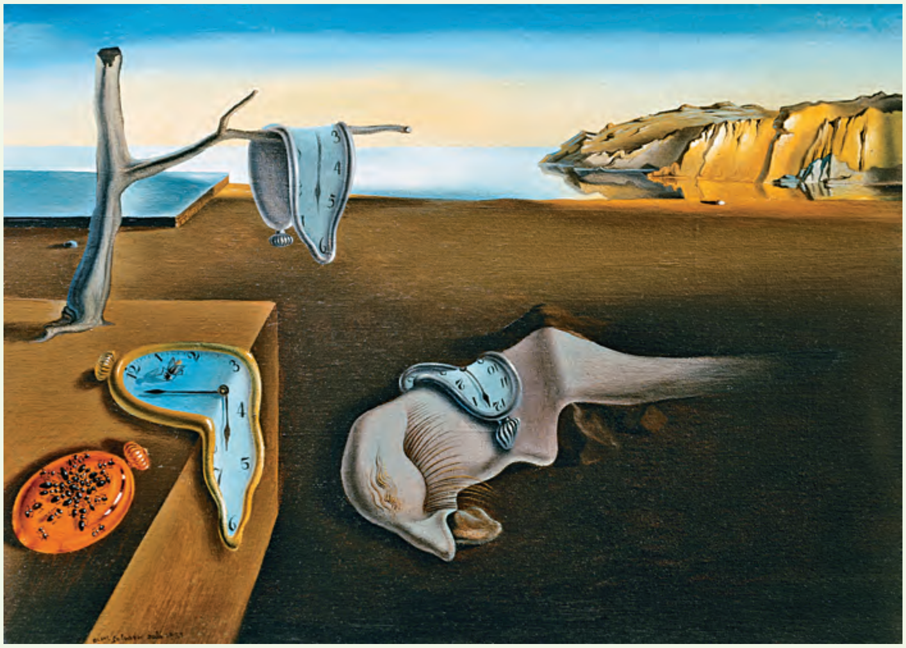
Simplify: $\sqrt[9]{x^3}$.

Solution

$$\sqrt[9]{x^3} = x^{\frac{3}{9}} = x^{\frac{1}{3}} = \sqrt[3]{x}$$

Check Point 13 Simplify: $\sqrt[6]{x^3}$.

A Radical Idea: Time Is Relative



Salvador Dali "The Persistence of Memory" 1931, oil on canvas, $9\frac{1}{2} \times 13$ in. (24.1×33 cm). The Museum of Modern Art/Licensed by Scala-Art Resource, N.Y. © 1999 Demart Pro Arte, Geneva/Artists Rights Society (ARS), New York.

What does travel in space have to do with radicals? Imagine that in the future we will be able to travel at velocities approaching the speed of light (approximately 186,000 miles per second). According to Einstein's theory of special relativity, time would pass more quickly on Earth than it would in the moving spaceship. The special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth. In this formula, v is the astronaut's speed and c is the speed of light. As the astronaut's speed approaches the speed of light, we can substitute c for v.

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$$ Einstein's equation gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth.

$$R_a = R_f\sqrt{1 - \left(\frac{c}{c}\right)^2}$$ The velocity, v, is approaching the speed of light, c, so let $v = c$.

$$= R_f\sqrt{1 - 1}$$ $\left(\frac{c}{c}\right)^2 = 1^2 = 1 \cdot 1 = 1$

$$= R_f\sqrt{0}$$ Simplify the radicand: $1 - 1 = 0$.

$$= R_f \cdot 0$$ $\sqrt{0} = 0$

$$= 0$$ Multiply: $R_f \cdot 0 = 0$.

Close to the speed of light, the astronaut's aging rate, R_a, relative to a friend, R_f, on Earth is nearly 0. What does this mean? As we age here on Earth, the space traveler would barely get older. The space traveler would return to an unknown futuristic world in which friends and loved ones would be long gone.

Exercise Set P.3

Practice Exercises

Evaluate each expression in Exercises 1–12, or indicate that the root is not a real number.

1. $\sqrt{36}$
2. $\sqrt{25}$
3. $-\sqrt{36}$
4. $-\sqrt{25}$
5. $\sqrt{-36}$
6. $\sqrt{-25}$
7. $\sqrt{25-16}$
8. $\sqrt{144+25}$
9. $\sqrt{25}-\sqrt{16}$
10. $\sqrt{144}+\sqrt{25}$
11. $\sqrt{(-13)^2}$
12. $\sqrt{(-17)^2}$

Use the product rule to simplify the expressions in Exercises 13–22. In Exercises 17–22, assume that variables represent nonnegative real numbers.

13. $\sqrt{50}$
14. $\sqrt{27}$
15. $\sqrt{45x^2}$
16. $\sqrt{125x^2}$
17. $\sqrt{2x}\cdot\sqrt{6x}$
18. $\sqrt{10x}\cdot\sqrt{8x}$
19. $\sqrt{x^3}$
20. $\sqrt{y^3}$
21. $\sqrt{2x^2}\cdot\sqrt{6x}$
22. $\sqrt{6x}\cdot\sqrt{3x^2}$

Use the quotient rule to simplify the expressions in Exercises 23–32. Assume that $x > 0$.

23. $\sqrt{\frac{1}{81}}$
24. $\sqrt{\frac{1}{49}}$
25. $\sqrt{\frac{49}{16}}$
26. $\sqrt{\frac{121}{9}}$
27. $\frac{\sqrt{48x^3}}{\sqrt{3x}}$
28. $\frac{\sqrt{72x^3}}{\sqrt{8x}}$
29. $\frac{\sqrt{150x^4}}{\sqrt{3x}}$
30. $\frac{\sqrt{24x^4}}{\sqrt{3x}}$
31. $\frac{\sqrt{200x^3}}{\sqrt{10x^{-1}}}$
32. $\frac{\sqrt{500x^3}}{\sqrt{10x^{-1}}}$

In Exercises 33–44, add or subtract terms whenever possible.

33. $7\sqrt{3}+6\sqrt{3}$
34. $8\sqrt{5}+11\sqrt{5}$
35. $6\sqrt{17x}-8\sqrt{17x}$
36. $4\sqrt{13x}-6\sqrt{13x}$
37. $\sqrt{8}+3\sqrt{2}$
38. $\sqrt{20}+6\sqrt{5}$
39. $\sqrt{50x}-\sqrt{8x}$
40. $\sqrt{63x}-\sqrt{28x}$
41. $3\sqrt{18}+5\sqrt{50}$
42. $4\sqrt{12}-2\sqrt{75}$
43. $3\sqrt{8}-\sqrt{32}+3\sqrt{72}-\sqrt{75}$
44. $3\sqrt{54}-2\sqrt{24}-\sqrt{96}+4\sqrt{63}$

In Exercises 45–54, rationalize the denominator.

45. $\frac{1}{\sqrt{7}}$
46. $\frac{2}{\sqrt{10}}$
47. $\frac{\sqrt{2}}{\sqrt{5}}$
48. $\frac{\sqrt{7}}{\sqrt{3}}$
49. $\frac{13}{3+\sqrt{11}}$
50. $\frac{3}{3+\sqrt{7}}$
51. $\frac{7}{\sqrt{5}-2}$
52. $\frac{5}{\sqrt{3}-1}$
53. $\frac{6}{\sqrt{5}+\sqrt{3}}$
54. $\frac{11}{\sqrt{7}-\sqrt{3}}$

Evaluate each expression in Exercises 55–66, or indicate that the root is not a real number.

55. $\sqrt[3]{125}$
56. $\sqrt[3]{8}$
57. $\sqrt[3]{-8}$
58. $\sqrt[3]{-125}$
59. $\sqrt[4]{-16}$
60. $\sqrt[4]{-81}$
61. $\sqrt[4]{(-3)^4}$
62. $\sqrt[4]{(-2)^4}$
63. $\sqrt[5]{(-3)^5}$
64. $\sqrt[5]{(-2)^5}$
65. $\sqrt[5]{-\frac{1}{32}}$
66. $\sqrt[6]{\frac{1}{64}}$

Simplify the radical expressions in Exercises 67–74.

67. $\sqrt[3]{32}$
68. $\sqrt[3]{150}$
69. $\sqrt[3]{x^4}$
70. $\sqrt[3]{x^5}$
71. $\sqrt[3]{9}\cdot\sqrt[3]{6}$
72. $\sqrt[3]{12}\cdot\sqrt[3]{4}$
73. $\frac{\sqrt[5]{64x^6}}{\sqrt[5]{2x}}$
74. $\frac{\sqrt[4]{162x^5}}{\sqrt[4]{2x}}$

In Exercises 75–82, add or subtract terms whenever possible.

75. $4\sqrt[5]{2}+3\sqrt[5]{2}$
76. $6\sqrt[5]{3}+2\sqrt[5]{3}$
77. $5\sqrt[3]{16}+\sqrt[3]{54}$
78. $3\sqrt[3]{24}+\sqrt[3]{81}$
79. $\sqrt[3]{54xy^3}-y\sqrt[3]{128x}$
80. $\sqrt[3]{24xy^3}-y\sqrt[3]{81x}$
81. $\sqrt{2}+\sqrt[3]{8}$
82. $\sqrt{3}+\sqrt[3]{15}$

In Exercises 83–90, evaluate each expression without using a calculator.

83. $36^{\frac{1}{2}}$
84. $121^{\frac{1}{2}}$
85. $8^{\frac{1}{3}}$
86. $27^{\frac{1}{3}}$
87. $125^{\frac{2}{3}}$
88. $8^{\frac{2}{3}}$
89. $32^{-\frac{4}{5}}$
90. $16^{-\frac{5}{2}}$

In Exercises 91–100, simplify using properties of exponents.

91. $\left(7x^{\frac{1}{3}}\right)\left(2x^{\frac{1}{4}}\right)$
92. $\left(3x^{\frac{2}{3}}\right)\left(4x^{\frac{3}{4}}\right)$
93. $\frac{20x^{\frac{1}{2}}}{5x^{\frac{1}{4}}}$
94. $\frac{72x^{\frac{3}{4}}}{9x^{\frac{1}{3}}}$
95. $\left(x^{\frac{2}{3}}\right)^3$
96. $\left(x^{\frac{4}{5}}\right)^5$

97. $(25x^4y^6)^{\frac{1}{2}}$

98. $(125x^9y^6)^{\frac{1}{3}}$

99. $\dfrac{\left(3y^{\frac{1}{4}}\right)^3}{y^{\frac{1}{12}}}$

100. $\dfrac{\left(2y^{\frac{1}{5}}\right)^4}{y^{\frac{3}{10}}}$

In Exercises 101–108, simplify by reducing the index of the radical.

101. $\sqrt[4]{5^2}$

102. $\sqrt[4]{7^2}$

103. $\sqrt[3]{x^6}$

104. $\sqrt[4]{x^{12}}$

105. $\sqrt[6]{x^4}$

106. $\sqrt[9]{x^6}$

107. $\sqrt[9]{x^6y^3}$

108. $\sqrt[12]{x^4y^8}$

Practice Plus

In Exercises 109–110, evaluate each expression.

109. $\sqrt[3]{\sqrt[4]{16} + \sqrt{625}}$

110. $\sqrt[3]{\sqrt{\sqrt{169} + \sqrt{9}} + \sqrt{\sqrt[3]{1000} + \sqrt[3]{216}}}$

In Exercises 111–114, simplify each expression. Assume that all variables represent positive numbers.

111. $(49x^{-2}y^4)^{-\frac{1}{2}}\left(xy^{\frac{1}{2}}\right)$

112. $(8x^{-6}y^3)^{\frac{1}{3}}\left(x^{\frac{5}{6}}y^{-\frac{1}{3}}\right)^6$

113. $\left(\dfrac{x^{-\frac{5}{4}}y^{\frac{1}{3}}}{x^{-\frac{3}{4}}}\right)^{-6}$

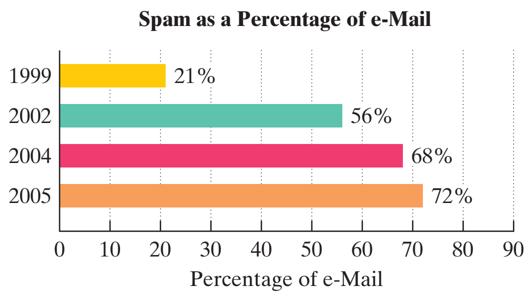
114. $\left(\dfrac{x^{\frac{1}{2}}y^{-\frac{7}{4}}}{y^{-\frac{5}{4}}}\right)^{-4}$

Application Exercises

115. The bar graph shows spam as a percentage of e-mail for four selected years. The data can be modeled by

$$y = 20.8\sqrt{x} + 21,$$

where y is the percentage of e-mail that is spam and x is the number of years after 1999.

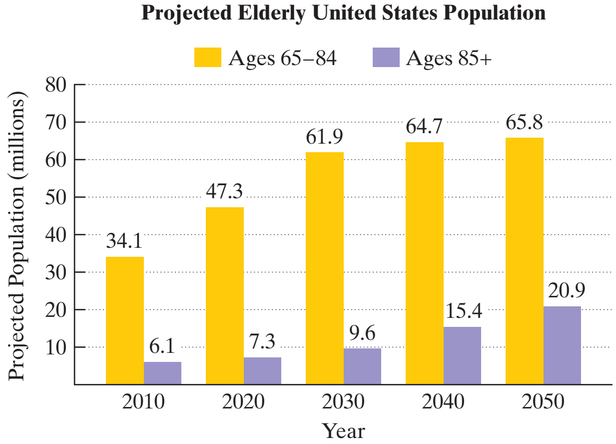


Source: IronPort

a. According to the model, what percentage of e-mail was spam in 2004? Use a calculator and round to the nearest tenth of a percent. Does this underestimate or overestimate the actual percentage given in the bar graph? By how much?

b. According to the model, what percentage of e-mail will be spam in 2011? Round to the nearest tenth of a percent.

116. America is getting older. The graph shows the projected elderly U.S. population for ages 65–84 and for ages 85 and older.



Source: U.S. Census Bureau

The formula $E = 5\sqrt{x} + 34.1$ models the projected number of elderly Americans ages 65–84, E, in millions, x years after 2010.

a. Use the formula to find the projected increase in the number of Americans ages 65–84, in millions, from 2020 to 2050. Express this difference in simplified radical form.

b. Use a calculator and write your answer in part (a) to the nearest tenth. Does this rounded decimal overestimate or underestimate the difference in the projected data shown by the bar graph? By how much?

117. The early Greeks believed that the most pleasing of all rectangles were **golden rectangles**, whose ratio of width to height is

$$\frac{w}{h} = \frac{2}{\sqrt{5} - 1}.$$

The Parthenon at Athens fits into a golden rectangle once the triangular pediment is reconstructed.

Rationalize the denominator of the golden ratio. Then use a calculator and find the ratio of width to height, correct to the nearest hundredth, in golden rectangles.

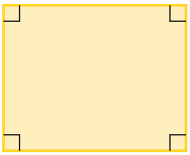
118. Use Einstein's special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2},$$

described in the essay on page 42, to solve this exercise. You are moving at 90% of the speed of light. Substitute $0.9c$ for v, your velocity, in the equation. What is your aging rate, correct to two decimal places, relative to a friend on Earth? If you are gone for 44 weeks, approximately how many weeks have passed for your friend?

The perimeter, P, of a rectangle with length l and width w is given by the formula $P = 2l + 2w$. The area, A, is given by the formula $A = lw$. In Exercises 119–120, use these formulas to find the perimeter and area of each rectangle. Express answers in simplified radical form. Remember that perimeter is measured in linear units, such as feet or meters, and area is measured in square units, such as square feet, ft^2, or square meters, m^2.

119.



120.

Writing in Mathematics

121. Explain how to simplify $\sqrt{10} \cdot \sqrt{5}$.

122. Explain how to add $\sqrt{3} + \sqrt{12}$.

123. Describe what it means to rationalize a denominator. Use both $\dfrac{1}{\sqrt{5}}$ and $\dfrac{1}{5 + \sqrt{5}}$ in your explanation.

124. What difference is there in simplifying $\sqrt[3]{(-5)^3}$ and $\sqrt[4]{(-5)^4}$?

125. What does $a^{\frac{m}{n}}$ mean?

126. Describe the kinds of numbers that have rational fifth roots.

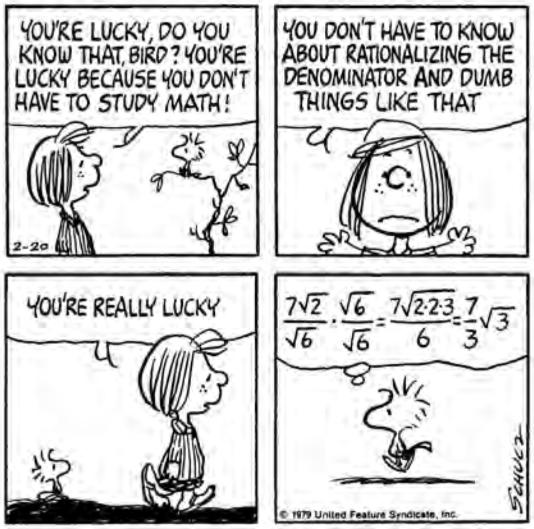
127. Why must a and b represent nonnegative numbers when we write $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$? Is it necessary to use this restriction in the case of $\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$? Explain.

128. Read the essay on page 42. The future is now: You have the opportunity to explore the cosmos in a starship traveling near the speed of light. The experience will enable you to understand the mysteries of the universe in deeply personal ways, transporting you to unimagined levels of knowing and being. The down side: You return from your two-year journey to a futuristic world in which friends and loved ones are long gone. Do you explore space or stay here on Earth? What are the reasons for your choice?

Critical Thinking Exercises

Make Sense? *In Exercises 129–132, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

129. The joke in this Peanuts cartoon would be more effective if Woodstock had rationalized the denominator correctly in the last frame.



130. Using my calculator, I determined that $6^7 = 279{,}936$, so 6 must be a seventh root of 279,936.

131. I simplified the terms of $2\sqrt{20} + 4\sqrt{75}$, and then I was able to add the like radicals.

132. When I use the definition for $a^{\frac{m}{n}}$, I usually prefer to first raise a to the m power because smaller numbers are involved.

In Exercises 133–136, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

133. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 49$

134. $8^{-\frac{1}{3}} = -2$

135. The cube root of -8 is not a real number.

136. $\dfrac{\sqrt{20}}{8} = \dfrac{\sqrt{10}}{4}$

In Exercises 137–138, fill in each box to make the statement true.

137. $(5 + \sqrt{\square})(5 - \sqrt{\square}) = 22$

138. $\sqrt{\square x^{\square}} = 5x^7$

139. Find exact value of $\sqrt{13 + \sqrt{2} + \dfrac{7}{3 + \sqrt{2}}}$ without the use of a calculator.

140. Place the correct symbol, $>$ or $<$, in the shaded area between the given numbers. *Do not use a calculator*. Then check your result with a calculator.

a. $3^{\frac{1}{2}} \; \square \; 3^{\frac{1}{3}}$

b. $\sqrt{7} + \sqrt{18} \; \square \; \sqrt{7 + 18}$

141. a. A mathematics professor recently purchased a birthday cake for her son with the inscription

$$\text{Happy } \left(2^{\frac{5}{2}} \cdot 2^{\frac{3}{4}} \div 2^{\frac{1}{4}}\right)\text{th Birthday.}$$

How old is the son?

b. The birthday boy, excited by the inscription on the cake, tried to wolf down the whole thing. Professor Mom, concerned about the possible metamorphosis of her son into a blimp, exclaimed, "Hold on! It is your birthday, so why not take $\dfrac{8^{-\frac{4}{3}} + 2^{-2}}{16^{-\frac{3}{4}} + 2^{-1}}$ of the cake? I'll eat half of what's left over." How much of the cake did the professor eat?

Preview Exercises

Exercises 142–144 will help you prepare for the material covered in the next section.

142. Multiply: $(2x^3y^2)(5x^4y^7)$.

143. Use the distributive property to multiply: $2x^4(8x^4 + 3x)$.

144. Simplify and express the answer in descending powers of x:

$$2x(x^2 + 4x + 5) + 3(x^2 + 4x + 5).$$