111. When performing the division

$$\frac{7x}{x+3} \div \frac{(x+3)^2}{x-5},$$

I began by dividing the numerator and the denominator by the common factor, $x + 3$.

112. I subtracted $\frac{3x-5}{x-1}$ from $\frac{x-3}{x-1}$ and obtained a constant.

In Exercises 113–116, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

113. $\frac{x^2-25}{x-5} = x - 5$

114. The expression $\frac{-3y-6}{y+2}$ simplifies to the consecutive integer that follows -4.

115. $\frac{2x-1}{x-7} + \frac{3x-1}{x-7} - \frac{5x-2}{x-7} = 0$

116. $6 + \frac{1}{x} = \frac{7}{x}$

In Exercises 117–119, perform the indicated operations.

117. $\frac{1}{x^n-1} - \frac{1}{x^n+1} - \frac{1}{x^{2n}-1}$

118. $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{x+3}\right)$

119. $(x-y)^{-1} + (x-y)^{-2}$

120. In one short sentence, five words or less, explain what

$$\frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}}$$

does to each number x.

Preview Exercises

Exercises 121–123 will help you prepare for the material covered in the next section.

121. If 6 is substituted for x in the equation

$$2(x-3) - 17 = 13 - 3(x+2),$$

is the resulting statement true or false?

122. Multiply and simplify: $12\left(\frac{x+2}{4} - \frac{x-1}{3}\right)$.

123. Evaluate

$$\frac{-b-\sqrt{b^2-4ac}}{2a}$$

for $a = 2$, $b = 9$, and $c = -5$.

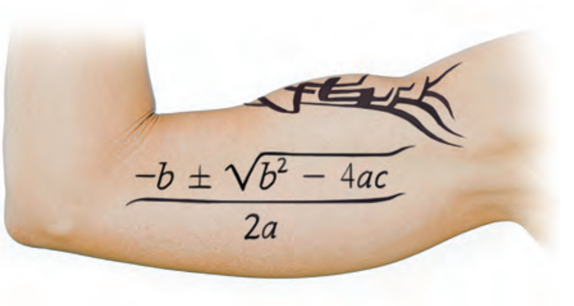
Section P.7 Equations

Objectives

1. Solve linear equations in one variable.
2. Solve linear equations containing fractions.
3. Solve rational equations with variables in the denominators.
4. Solve a formula for a variable.
5. Solve equations involving absolute value.
6. Solve quadratic equations by factoring.
7. Solve quadratic equations by the square root property.
8. Solve quadratic equations by completing the square.
9. Solve quadratic equations using the quadratic formula.
10. Use the discriminant to determine the number and type of solutions of quadratic equations.
11. Determine the most efficient method to use when solving a quadratic equation.
12. Solve radical equations.

Math tattoos. Who knew? Do you recognize the significance of this tattoo? The algebraic expression gives the solutions of a *quadratic equation*.

In this section, we will review how to solve a variety of equations, including linear equations, quadratic equations, and radical equations.



Solving Linear Equations in One Variable

We begin with a general definition of a linear equation in one variable.

Definition of a Linear Equation

A **linear equation in one variable x** is an equation that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers, and $a \neq 0$.

An example of a linear equation in one variable is

$$4x + 12 = 0.$$

Solving an equation in x involves determining all values of x that result in a true statement when substituted into the equation. Such values are **solutions**, or **roots**, of the equation. For example, substitute -3 for x in $4x + 12 = 0$. We obtain

$$4(-3) + 12 = 0, \quad \text{or} \quad -12 + 12 = 0.$$

This simplifies to the true statement $0 = 0$. Thus, -3 is a solution of the equation $4x + 12 = 0$. We also say that -3 **satisfies** the equation $4x + 12 = 0$, because when we substitute -3 for x, a true statement results. The set of all such solutions is called the equation's **solution set**. For example, the solution set of the equation $4x + 12 = 0$ is $\{-3\}$ because -3 is the equation's only solution.

Two or more equations that have the same solution set are called **equivalent equations**. For example, the equations

$$4x + 12 = 0 \quad \text{and} \quad 4x = -12 \quad \text{and} \quad x = -3$$

are equivalent equations because the solution set for each is $\{-3\}$. To solve a linear equation in x, we transform the equation into an equivalent equation one or more times. Our final equivalent equation should be of the form

$$x = \text{a number}.$$

The solution set of this equation is the set consisting of the number.

To generate equivalent equations, we will use the following principles:

Generating Equivalent Equations

An equation can be transformed into an equivalent equation by one or more of the following operations:

	Example
1. Simplify an expression by removing grouping symbols and combining like terms.	• $3(x - 6) = 6x - x$ $3x - 18 = 5x$
2. Add (or subtract) the same real number or variable expression on *both* sides of the equation.	• $3x - 18 = 5x$ $3x - 18 - 3x = 5x - 3x$ — Subtract $3x$ from both sides of the equation. $-18 = 2x$
3. Multiply (or divide) by the same *nonzero* quantity on *both* sides of the equation.	• $-18 = 2x$ $\frac{-18}{2} = \frac{2x}{2}$ — Divide both sides of the equation by 2. $-9 = x$
4. Interchange the two sides of the equation.	• $-9 = x$ $x = -9$

If you look closely at the equations in the box, you will notice that we have solved the equation $3(x - 6) = 6x - x$. The final equation, $x = -9$, with x isolated on the left side, shows that $\{-9\}$ is the solution set. The idea in solving a linear equation is to get the variable by itself on one side of the equal sign and a number by itself on the other side.

1 Solve linear equations in one variable.

Here is a step-by-step procedure for solving a linear equation in one variable. Not all of these steps are necessary to solve every equation.

Solving a Linear Equation

1. Simplify the algebraic expression on each side by removing grouping symbols and combining like terms.
2. Collect all the variable terms on one side and all the numbers, or constant terms, on the other side.
3. Isolate the variable and solve.
4. Check the proposed solution in the original equation.

EXAMPLE 1 Solving a Linear Equation

Solve and check: $2(x - 3) - 17 = 13 - 3(x + 2)$.

Solution

Step 1 Simplify the algebraic expression on each side.

Do not begin with 13 – 3. Multiplication (the distributive property) is applied before subtraction.

$$2(x - 3) - 17 = 13 - 3(x + 2) \quad \text{This is the given equation.}$$

$$2x - 6 - 17 = 13 - 3x - 6 \quad \text{Use the distributive property.}$$

$$2x - 23 = -3x + 7 \quad \text{Combine like terms.}$$

Step 2 Collect variable terms on one side and constant terms on the other side. We will collect variable terms on the left by adding $3x$ to both sides. We will collect the numbers on the right by adding 23 to both sides.

$$2x - 23 + 3x = -3x + 7 + 3x \quad \text{Add } 3x \text{ to both sides.}$$

$$5x - 23 = 7 \quad \text{Simplify: } 2x + 3x = 5x.$$

$$5x - 23 + 23 = 7 + 23 \quad \text{Add 23 to both sides.}$$

$$5x = 30 \quad \text{Simplify.}$$

Step 3 Isolate the variable and solve. We isolate the variable, x, by dividing both sides of $5x = 30$ by 5.

$$\frac{5x}{5} = \frac{30}{5} \quad \text{Divide both sides by 5.}$$

$$x = 6 \quad \text{Simplify.}$$

Step 4 Check the proposed solution in the original equation. Substitute 6 for x in the original equation.

$$2(x - 3) - 17 = 13 - 3(x + 2) \quad \text{This is the original equation.}$$

$$2(6 - 3) - 17 \stackrel{?}{=} 13 - 3(6 + 2) \quad \text{Substitute 6 for } x.$$

$$2(3) - 17 \stackrel{?}{=} 13 - 3(8) \quad \text{Simplify inside parentheses.}$$

$$6 - 17 \stackrel{?}{=} 13 - 24 \quad \text{Multiply.}$$

$$-11 = -11 \quad \text{Subtract.}$$

The true statement $-11 = -11$ verifies that the solution set is $\{6\}$.

Discovery

Solve the equation in Example 1 by collecting terms with the variable on the right and numerical terms on the left. What do you observe?

Check Point 1 Solve and check: $4(2x + 1) = 29 + 3(2x - 5)$.

2 Solve linear equations containing fractions.

Linear Equations with Fractions

Equations are easier to solve when they do not contain fractions. How do we remove fractions from an equation? We begin by multiplying both sides of the equation by the least common denominator of any fractions in the equation. The least common denominator is the smallest number that all denominators will divide into. Multiplying every term on both sides of the equation by the least common denominator will eliminate the fractions in the equation. Example 2 shows how we "clear an equation of fractions."

EXAMPLE 2 Solving a Linear Equation Involving Fractions

Solve and check: $\frac{x+2}{4} - \frac{x-1}{3} = 2.$

Solution The fractional terms have denominators of 4 and 3. The smallest number that is divisible by 4 and 3 is 12. We begin by multiplying both sides of the equation by 12, the least common denominator.

$$\frac{x+2}{4} - \frac{x-1}{3} = 2$$ This is the given equation.

$$12\left(\frac{x+2}{4} - \frac{x-1}{3}\right) = 12 \cdot 2$$ Multiply both sides by 12.

$$12\left(\frac{x+2}{4}\right) - 12\left(\frac{x-1}{3}\right) = 24$$ Use the distributive property and multiply each term on the left by 12.

$$\overset{3}{\cancel{12}}\left(\frac{x+2}{\underset{1}{\cancel{4}}}\right) - \overset{4}{\cancel{12}}\left(\frac{x-1}{\underset{1}{\cancel{3}}}\right) = 24$$ Divide out common factors in each multiplication on the left.

$$3(x+2) - 4(x-1) = 24$$ The fractions are now cleared.

$$3x + 6 - 4x + 4 = 24$$ Use the distributive property.

$$-x + 10 = 24$$ Combine like terms: $3x - 4x = -x$ and $6 + 4 = 10$.

$$-x + 10 - 10 = 24 - 10$$ Subtract 10 from both sides.

$$-x = 14$$ Simplify.

We're not finished. A negative sign should not precede the variable.

Isolate x by multiplying or dividing both sides of this equation by -1.

$$\frac{-x}{-1} = \frac{14}{-1}$$ Divide both sides by -1.

$$x = -14$$ Simplify.

Check the proposed solution. Substitute -14 for x in the original equation. You should obtain $2 = 2$. This true statement verifies that the solution set is $\{-14\}$.

Check Point 2 Solve and check: $\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}.$

3 Solve rational equations with variables in the denominators.

Rational Equations

A **rational equation** is an equation containing one or more rational expressions. In Example 2, we solved a rational equation with constants in the denominators. This rational equation was a linear equation. Now, let's consider a rational equation such as

$$\frac{3}{x+6}+\frac{1}{x-2}=\frac{4}{x^2+4x-12}.$$

Can you see how this rational equation differs from the rational equation that we solved earlier? The variable appears in the denominators. Although this rational equation is not a linear equation, the solution procedure still involves multiplying each side by the least common denominator. However, we must avoid any values of the variable that make a denominator zero.

EXAMPLE 3 Solving a Rational Equation

Solve: $\dfrac{3}{x+6}+\dfrac{1}{x-2}=\dfrac{4}{x^2+4x-12}.$

Solution To identify values of x that make denominators zero, let's factor $x^2+4x-12$, the denominator on the right. This factorization is also necessary in identifying the least common denominator.

$$\frac{3}{x+6}+\frac{1}{x-2}=\frac{4}{(x+6)(x-2)}$$

This denominator is zero if $x=-6$. This denominator is zero if $x=2$. This denominator is zero if $x=-6$ or $x=2$.

We see that x cannot equal -6 or 2. The least common denominator is $(x+6)(x-2)$.

$$\frac{3}{x+6}+\frac{1}{x-2}=\frac{4}{(x+6)(x-2)},\quad x\neq-6,\ x\neq 2$$

This is the given equation with a denominator factored.

$$(x+6)(x-2)\left(\frac{3}{x+6}+\frac{1}{x-2}\right)=(x+6)(x-2)\cdot\frac{4}{(x+6)(x-2)}$$

Multiply both sides by $(x+6)(x-2)$, the LCD.

$$\cancel{(x+6)}(x-2)\cdot\frac{3}{\cancel{x+6}}+(x+6)\cancel{(x-2)}\cdot\frac{1}{\cancel{x-2}}=\cancel{(x+6)}\cancel{(x-2)}\cdot\frac{4}{\cancel{(x+6)}\cancel{(x-2)}}$$

Use the distributive property and divide out common factors.

$$3(x-2)+1(x+6)=4$$

Simplify. This equation is cleared of fractions.

$$3x-6+x+6=4$$

Use the distributive property.

$$4x=4$$

Combine like terms.

$$\frac{4x}{4}=\frac{4}{4}$$

Divide both sides by 4.

$$x=1$$

Simplify. This is not part of the restriction that $x\neq-6$ and $x\neq 2$.

Check the proposed solution. Substitute 1 for x in the original equation. You should obtain $-\frac{4}{7}=-\frac{4}{7}$. This true statement verifies that the solution set is $\{1\}$.

Check Point 3 Solve: $\dfrac{6}{x+3}-\dfrac{5}{x-2}=\dfrac{-20}{x^2+x-6}.$

EXAMPLE 4 Solving a Rational Equation

Solve: $\dfrac{1}{x+1} = \dfrac{2}{x^2-1} - \dfrac{1}{x-1}$.

Solution We begin by factoring $x^2 - 1$.

$$\frac{1}{x+1} = \frac{2}{(x+1)(x-1)} - \frac{1}{x-1}$$

This denominator is zero if $x = -1$. This denominator is zero if $x = -1$ or $x = 1$. This denominator is zero if $x = 1$.

We see that x cannot equal -1 or 1. The least common denominator is $(x+1)(x-1)$.

$$\frac{1}{x+1} = \frac{2}{(x+1)(x-1)} - \frac{1}{x-1}, \quad x \neq -1, x \neq 1$$

This is the given equation with a denominator factored.

$$(x+1)(x-1)\cdot\frac{1}{x+1} = (x+1)(x-1)\left(\frac{2}{(x+1)(x-1)} - \frac{1}{x-1}\right)$$

Multiply both sides by $(x+1)(x-1)$, the LCD.

$$\cancel{(x+1)}(x-1)\cdot\frac{1}{\cancel{x+1}} = \cancel{(x+1)}\,\cancel{(x-1)}\cdot\frac{2}{\cancel{(x+1)}\,\cancel{(x-1)}} - (x+1)\cancel{(x-1)}\cdot\frac{1}{\cancel{(x-1)}}$$

Use the distributive property and divide out common factors.

$$1(x-1) = 2 - (x+1)$$ Simplify. This equation is cleared of fractions.

$$x - 1 = 2 - x - 1$$ Simplify.

$$x - 1 = -x + 1$$ Combine numerical terms.

$$x + x - 1 = -x + x + 1$$ Add x to both sides.

$$2x - 1 = 1$$ Simplify.

$$2x - 1 + 1 = 1 + 1$$ Add 1 to both sides.

$$2x = 2$$ Simplify.

$$\frac{2x}{2} = \frac{2}{2}$$ Divide both sides by 2.

$$x = 1$$ Simplify.

The proposed solution, 1, is *not* a solution because of the restriction that $x \neq 1$. There is *no solution to this equation*. The solution set for this equation contains no elements. The solution set is $\varnothing$, the empty set.

Study Tip

Reject any proposed solution that causes any denominator in an equation to equal 0.

Check Point 4 Solve: $\dfrac{1}{x+2} = \dfrac{4}{x^2-4} - \dfrac{1}{x-2}$.

4 Solve a formula for a variable.

Solving a Formula for One of Its Variables

Solving a formula for a variable means rewriting the formula so that the variable is isolated on one side of the equation. It does not mean obtaining a numerical value for that variable.

To solve a formula for one of its variables, treat that variable as if it were the only variable in the equation. Think of the other variables as if they were numbers.

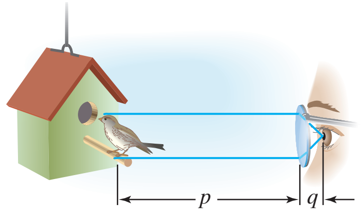


Figure P.12

EXAMPLE 5 Solving a Formula for a Variable

If you wear glasses, did you know that each lens has a measurement called its focal length, f? When an object is in focus, its distance from the lens, p, and the distance from the lens to your retina, q, satisfy the formula

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

(See **Figure P.12**.) Solve this formula for p.

Solution Our goal is to isolate the variable p. We begin by multiplying both sides by the least common denominator, pqf, to clear the equation of fractions.

We need to isolate p.

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$ This is the given formula.

$$pqf\left(\frac{1}{p} + \frac{1}{q}\right) = pqf\left(\frac{1}{f}\right)$$ Multiply both sides by pqf, the LCD.

$$\cancel{p}qf\left(\frac{1}{\cancel{p}}\right) + p\cancel{q}f\left(\frac{1}{\cancel{q}}\right) = pq\cancel{f}\left(\frac{1}{\cancel{f}}\right)$$ Use the distributive property on the left side and divide out common factors.

$$qf + pf = pq$$ Simplify. The formula is cleared of fractions.

We need to isolate p.

To collect terms with p on one side of the equation, subtract pf from both sides. Then factor p from the two resulting terms on the right to convert two occurrences of p into one.

$$qf + pf = pq$$ This is the equation cleared of fractions.

$$qf + pf - pf = pq - pf$$ Subtract pf from both sides.

$$qf = pq - pf$$ Simplify.

$$qf = p(q - f)$$ Factor out p, the specified variable.

$$\frac{qf}{q-f} = \frac{p(q-f)}{q-f}$$ Divide both sides by $q - f$ and solve for p.

$$\frac{qf}{q-f} = p$$ Simplify.

Study Tip

You cannot solve $qf + pf = pq$ for p by dividing both sides by q and writing

$$\frac{qf + pf}{q} = p.$$

When a formula is solved for a specified variable, that variable must be isolated on one side. The variable p occurs on both sides of

$$\frac{qf + pf}{q} = p.$$

Check Point 5 Solve for q: $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$.

5 Solve equations involving absolute value.

Equations Involving Absolute Value

We have seen that the absolute value of x, denoted $|x|$, describes the distance of x from zero on a number line. Now consider an **absolute value equation**, such as

$$|x| = 2.$$

This means that we must determine real numbers whose distance from the origin on a number line is 2. **Figure P.13** shows that there are two numbers such that $|x| = 2$, namely, 2 and -2. We write $x = 2$ or $x = -2$. This observation can be generalized as follows:

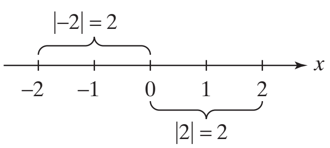


Figure P.13

Rewriting an Absolute Value Equation without Absolute Value Bars

If c is a positive real number and u represents any algebraic expression, then $|u| = c$ is equivalent to $u = c$ or $u = -c$.

EXAMPLE 6 Solving an Equation Involving Absolute Value

Solve: $5|1 - 4x| - 15 = 0$.

Solution

$5|1 - 4x| - 15 = 0$ This is the given equation.

We need to isolate $|1 - 4x|$, the absolute value expression.

$5|1 - 4x| = 15$ Add 15 to both sides.

$|1 - 4x| = 3$ Divide both sides by 5.

$1 - 4x = 3$ or $1 - 4x = -3$ Rewrite $|u| = c$ as $u = c$ or $u = -c$.

$-4x = 2$ $\quad$ $-4x = -4$ Subtract 1 from both sides of each equation.

$x = -\frac{1}{2}$ $\quad$ $x = 1$ Divide both sides of each equation by -4.

Take a moment to check $-\frac{1}{2}$ and 1, the proposed solutions, in the original equation, $5|1 - 4x| - 15 = 0$. In each case, you should obtain the true statement $0 = 0$. The solution set is $\left\{-\frac{1}{2}, 1\right\}$.

Check Point 6 Solve: $4|1 - 2x| - 20 = 0$.

The absolute value of a number is never negative. Thus, if u is an algebraic expression and c is a negative number, then $|u| = c$ has no solution. For example, the equation $|3x - 6| = -2$ has no solution because $|3x - 6|$ cannot be negative. The solution set is $\varnothing$, the empty set.

The absolute value of 0 is 0. Thus, if u is an algebraic expression and $|u| = 0$, the solution is found by solving $u = 0$. For example, the solution of $|x - 2| = 0$ is obtained by solving $x - 2 = 0$. The solution is 2 and the solution set is $\{2\}$.

6 Solve quadratic equations by factoring.

Quadratic Equations and Factoring

Linear equations are first-degree polynomial equations of the form $ax + b = 0$. *Quadratic equations* are second-degree polynomial equations and contain an additional term involving the square of the variable.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **general form**

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$. A quadratic equation in x is also called a **second-degree polynomial equation** in x.

Here are examples of quadratic equations in general form:

$$4x^2 - 2x \quad = 0 \qquad\qquad 2x^2 + 7x - 4 = 0.$$

$a = 4$ $\quad$ $b = -2$ $\quad$ $c = 0$ $\qquad$ $a = 2$ $\quad$ $b = 7$ $\quad$ $c = -4$

Some quadratic equations, including the two shown above, can be solved by factoring and using the **zero-product principle**.

The Zero-Product Principle

If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero.

$$\text{If } AB = 0, \text{ then } A = 0 \text{ or } B = 0.$$

The zero-product principle can be applied only when a quadratic equation is in general form, with zero on one side of the equation.

Solving a Quadratic Equation by Factoring

1. If necessary, rewrite the equation in the general form $ax^2 + bx + c = 0$, moving all terms to one side, thereby obtaining zero on the other side.
2. Factor completely.
3. Apply the zero-product principle, setting each factor containing a variable equal to zero.
4. Solve the equations in step 3.
5. Check the solutions in the original equation.

EXAMPLE 7 Solving Quadratic Equations by Factoring

Solve by factoring:

a. $4x^2 - 2x = 0$ **b.** $2x^2 + 7x = 4$.

Solution

a.

$4x^2 - 2x = 0$ — The given equation is in general form, with zero on one side.

$2x(2x - 1) = 0$ — Factor.

$2x = 0 \quad \text{or} \quad 2x - 1 = 0$ — Use the zero-product principle and set each factor equal to zero.

$x = 0 \qquad 2x = 1$ — Solve the resulting equations.

$x = \frac{1}{2}$

Check the proposed solutions, 0 and $\frac{1}{2}$, in the original equation.

Check 0:

$$4x^2 - 2x = 0$$
$$4 \cdot 0^2 - 2 \cdot 0 \stackrel{?}{=} 0$$
$$0 - 0 \stackrel{?}{=} 0$$
$$0 = 0, \quad \text{true}$$

Check $\frac{1}{2}$:

$$4x^2 - 2x = 0$$
$$4\left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{1}{2}\right) \stackrel{?}{=} 0$$
$$4\left(\tfrac{1}{4}\right) - 2\left(\tfrac{1}{2}\right) \stackrel{?}{=} 0$$
$$1 - 1 \stackrel{?}{=} 0$$
$$0 = 0, \quad \text{true}$$

The solution set is $\left\{0, \frac{1}{2}\right\}$.

b.

$2x^2 + 7x = 4$ — This is the given equation.

$2x^2 + 7x - 4 = 4 - 4$ — Subtract 4 from both sides and write the quadratic equation in general form.

$2x^2 + 7x - 4 = 0$ — Simplify.

$(2x - 1)(x + 4) = 0$ — Factor.

$2x - 1 = 0 \quad \text{or} \quad x + 4 = 0$ — Use the zero-product principle and set each factor equal to zero.

$2x = 1 \qquad x = -4$ — Solve the resulting equations.

$x = \frac{1}{2}$

Check the proposed solutions, $\frac{1}{2}$ and -4, in the original equation.

Check $\frac{1}{2}$:

$$2x^2 + 7x = 4$$
$$2\left(\tfrac{1}{2}\right)^2 + 7\left(\tfrac{1}{2}\right) \stackrel{?}{=} 4$$
$$\tfrac{1}{2} + \tfrac{7}{2} \stackrel{?}{=} 4$$
$$4 = 4, \text{ true}$$

Check -4:

$$2x^2 + 7x = 4$$
$$2(-4)^2 + 7(-4) \stackrel{?}{=} 4$$
$$32 + (-28) \stackrel{?}{=} 4$$
$$4 = 4, \text{ true}$$

The solution set is $\left\{-4, \frac{1}{2}\right\}$.

Check Point 7 Solve by factoring:

a. $3x^2 - 9x = 0$ **b.** $2x^2 + x = 1$.

7 Solve quadratic equations by the square root property.

Quadratic Equations and the Square Root Property

Quadratic equations of the form $u^2 = d$, where u is an algebraic expression and d is a nonzero real number, can be solved by the *square root property*. First, isolate the squared expression u^2 on one side of the equation and the number d on the other side. Then take the square root of both sides. Remember, there are two numbers whose square is d. One number is $\sqrt{d}$ and one is $-\sqrt{d}$.

We can use factoring to verify that $u^2 = d$ has these two solutions.

$$u^2 = d \quad \text{This is the given equation.}$$
$$u^2 - d = 0 \quad \text{Move all terms to one side and obtain zero on the other side.}$$
$$\left(u + \sqrt{d}\right)\left(u - \sqrt{d}\right) = 0 \quad \text{Factor.}$$
$$u + \sqrt{d} = 0 \quad \text{or} \quad u - \sqrt{d} = 0 \quad \text{Set each factor equal to zero.}$$
$$u = -\sqrt{d} \qquad u = \sqrt{d} \quad \text{Solve the resulting equations.}$$

Because the solutions differ only in sign, we can write them in abbreviated notation as $u = \pm\sqrt{d}$. We read this as "u equals positive or negative square root of d" or "u equals plus or minus square root of d."

Now that we have verified these solutions, we can solve $u^2 = d$ directly by taking square roots. This process is called the **square root property**.

The Square Root Property

If u is an algebraic expression and d is a positive real number, then $u^2 = d$ has exactly two solutions:

$$\text{If } u^2 = d, \text{ then } u = \sqrt{d} \text{ or } u = -\sqrt{d}.$$

Equivalently,

$$\text{If } u^2 = d, \text{ then } u = \pm\sqrt{d}.$$

EXAMPLE 8 Solving Quadratic Equations by the Square Root Property

Solve by the square root property:

a. $3x^2 - 15 = 0$ **b.** $(x - 2)^2 = 6$.

Solution To apply the square root property, we need a squared expression by itself on one side of the equation.

$$3x^2 - 15 = 0 \qquad\qquad (x - 2)^2 = 6$$

We want x^2 by itself.

The squared expression is by itself.

a.

$3x^2 - 15 = 0$	This is the original equation.
$3x^2 = 15$	Add 15 to both sides.
$x^2 = 5$	Divide both sides by 3.
$x = \sqrt{5}$ or $x = -\sqrt{5}$	Apply the square root property. Equivalently, $x = \pm\sqrt{5}$.

By checking both proposed solutions in the original equation, we can confirm that the solution set is $\{-\sqrt{5}, \sqrt{5}\}$ or $\{\pm\sqrt{5}\}$.

b.

$(x-2)^2 = 6$	This is the original equation.
$x - 2 = \pm\sqrt{6}$	Apply the square root property.
$x = 2 \pm \sqrt{6}$	Add 2 to both sides.

By checking both values in the original equation, we can confirm that the solution set is $\{2 + \sqrt{6}, 2 - \sqrt{6}\}$ or $\{2 \pm \sqrt{6}\}$.

Check Point 8 Solve by the square root property:

a. $3x^2 - 21 = 0$ **b.** $(x+5)^2 = 11$.

8 Solve quadratic equations by completing the square.

Quadratic Equations and Completing the Square

How do we solve an equation in the form $ax^2 + bx + c = 0$ if the trinomial $ax^2 + bx + c$ cannot be factored? We cannot use the zero-product principle in such a case. However, we can convert the equation into an equivalent equation that can be solved using the square root property. This is accomplished by **completing the square**.

Completing the Square

If $x^2 + bx$ is a binomial, then by adding $\left(\frac{b}{2}\right)^2$, which is the square of half the coefficient of x, a perfect square trinomial will result. That is,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

We can solve any quadratic equation by completing the square. If the coefficient of the x^2-term is one, we add the square of half the coefficient of x to both sides of the equation. **When you add a constant term to one side of the equation to complete the square, be certain to add the same constant to the other side of the equation.** These ideas are illustrated in Example 9.

EXAMPLE 9 Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $x^2 - 6x + 4 = 0$.

Solution We begin by subtracting 4 from both sides. This is done to isolate the binomial $x^2 - 6x$ so that we can complete the square.

$x^2 - 6x + 4 = 0$	This is the original equation.
$x^2 - 6x = -4$	Subtract 4 from both sides.

Next, we work with $x^2 - 6x = -4$ and complete the square. Find half the coefficient of the x-term and square it. The coefficient of the x-term is -6. Half of -6 is -3 and $(-3)^2 = 9$. Thus, we add 9 to both sides of the equation.

$$x^2 - 6x + 9 = -4 + 9$$ Add 9 to both sides of $x^2 - 6x = -4$ to complete the square.

$$(x-3)^2 = 5$$ Factor and simplify.

$$x - 3 = \sqrt{5} \quad \text{or} \quad x - 3 = -\sqrt{5}$$ Apply the square root property.

$$x = 3 + \sqrt{5} \qquad x = 3 - \sqrt{5}$$ Add 3 to both sides in each equation.

The solutions are $3 \pm \sqrt{5}$ and the solution set is $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$, or $\{3 \pm \sqrt{5}\}$.

Check Point 9 Solve by completing the square: $x^2 + 4x - 1 = 0$.

9 Solve quadratic equations using the quadratic formula.

Quadratic Equations and the Quadratic Formula

We can use the method of completing the square to derive a formula that can be used to solve all quadratic equations. The derivation given here also shows a particular quadratic equation, $3x^2 - 2x - 4 = 0$, to specifically illustrate each of the steps.

Notice that if the coefficient of the x^2-term in a quadratic equation is not one, you must divide each side of the equation by this coefficient before completing the square.

Deriving the Quadratic Formula

General Form of a Quadratic Equation	Comment	A Specific Example
$ax^2 + bx + c = 0, a > 0$	This is the given equation.	$3x^2 - 2x - 4 = 0$
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide both sides by a so that the coefficient of x^2 is 1.	$x^2 - \frac{2}{3}x - \frac{4}{3} = 0$
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Isolate the binomial by adding $-\frac{c}{a}$ on both sides of the equation.	$x^2 - \frac{2}{3}x = \frac{4}{3}$
$x^2 + \underbrace{\frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{(\text{half})^2} = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$	Complete the square. Add the square of half the coefficient of x to both sides.	$x^2 - \underbrace{\frac{2}{3}x + \left(-\frac{1}{3}\right)^2}_{(\text{half})^2} = \frac{4}{3} + \left(-\frac{1}{3}\right)^2$ $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2}$	Factor on the left side and obtain a common denominator on the right side.	$\left(x - \frac{1}{3}\right)^2 = \frac{4}{3} \cdot \frac{3}{3} + \frac{1}{9}$
$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Add fractions on the right side.	$\left(x - \frac{1}{3}\right)^2 = \frac{12 + 1}{9}$ $\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}$
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Apply the square root property.	$x - \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Take the square root of the quotient, simplifying the denominator.	$x - \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Solve for x by subtracting $\frac{b}{2a}$ from both sides.	$x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Combine fractions on the right side.	$x = \frac{1 \pm \sqrt{13}}{3}$

The formula shown at the bottom of the left column is called the *quadratic formula*. A similar proof shows that the same formula can be used to solve quadratic equations if a, the coefficient of the x^2-term, is negative.

The Quadratic Formula

The solutions of a quadratic equation in general form $ax^2 + bx + c = 0$, with $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

x equals negative b plus or minus the square root of $b^2 - 4ac$, all divided by $2a$.

To use the quadratic formula, write the quadratic equation in general form if necessary. Then determine the numerical values for a (the coefficient of the x^2-term), b (the coefficient of the x-term), and c (the constant term). Substitute the values of a, b, and c into the quadratic formula and evaluate the expression. The $\pm$ sign indicates that there are two solutions of the equation.

EXAMPLE 10 Solving a Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula: $2x^2 - 6x + 1 = 0$.

Solution The given equation is in general form. Begin by identifying the values for a, b, and c.

$$2x^2 - 6x + 1 = 0$$

$a = 2$ $b = -6$ $c = 1$

Substituting these values into the quadratic formula and simplifying gives the equation's solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Use the quadratic formula.

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(1)}}{2 \cdot 2}$$ Substitute the values for a, b, and c: $a = 2$, $b = -6$, and $c = 1$.

$$= \frac{6 \pm \sqrt{36 - 8}}{4}$$ $-(-6) = 6$, $(-6)^2 = (-6)(-6) = 36$, and $4(2)(1) = 8$.

$$= \frac{6 \pm \sqrt{28}}{4}$$ Complete the subtraction under the radical.

$$= \frac{6 \pm 2\sqrt{7}}{4}$$ $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

$$= \frac{2(3 \pm \sqrt{7})}{4}$$ Factor out 2 from the numerator.

$$= \frac{3 \pm \sqrt{7}}{2}$$ Divide the numerator and denominator by 2.

The solution set is $\left\{\frac{3 + \sqrt{7}}{2}, \frac{3 - \sqrt{7}}{2}\right\}$ or $\left\{\frac{3 \pm \sqrt{7}}{2}\right\}$.

Study Tip

Checking irrational solutions can be time-consuming. The solutions given by the quadratic formula are always correct, unless you have made a careless error. Checking for computational errors or errors in simplification is sufficient.

Check Point 10 Solve using the quadratic formula:

$$2x^2 + 2x - 1 = 0.$$

10 Use the discriminant to determine the number and type of solutions of quadratic equations.

Quadratic Equations and the Discriminant

The quantity $b^2 - 4ac$, which appears under the radical sign in the quadratic formula, is called the **discriminant**. **Table P.4** shows how the discriminant of the quadratic equation $ax^2 + bx + c = 0$ determines the number and type of solutions.

Table P.4 The Discriminant and the Kinds of Solutions to $ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$	Kinds of Solutions to $ax^2 + bx + c = 0$
$b^2 - 4ac > 0$	**Two unequal real solutions:** If a, b, and c are rational numbers and the discriminant is a perfect square, the solutions are rational. If the discriminant is not a perfect square, the solutions are irrational.
$b^2 - 4ac = 0$	**One solution (a repeated solution) that is a real number:** If a, b, and c are rational numbers, the repeated solution is also a rational number.
$b^2 - 4ac < 0$	**No real solutions**

EXAMPLE 11 Using the Discriminant

Compute the discriminant of $4x^2 - 8x + 1 = 0$. What does the discriminant indicate about the number and type of solutions?

Solution Begin by identifying the values for a, b, and c.

$$2x^2 - 6x + 1 = 0$$

$a = 2$ $b = -6$ $c = 1$

Substitute and compute the discriminant:

$$b^2 - 4ac = (-8)^2 - 4 \cdot 4 \cdot 1 = 64 - 16 = 48.$$

The discriminant is 48. Because the discriminant is positive, the equation $4x^2 - 8x + 1 = 0$ has two unequal real solutions.

Check Point 11 Compute the discriminant of $3x^2 - 2x + 5 = 0$. What does the discriminant indicate about the number and type of solutions?

11 Determine the most efficient method to use when solving a quadratic equation.

Determining Which Method to Use

All quadratic equations can be solved by the quadratic formula. However, if an equation is in the form $u^2 = d$, such as $x^2 = 5$ or $(2x + 3)^2 = 8$, it is faster to use the square root property, taking the square root of both sides. If the equation is not in the form $u^2 = d$, write the quadratic equation in general form $(ax^2 + bx + c = 0)$. Try to solve the equation by factoring. If $ax^2 + bx + c$ cannot be factored, then solve the quadratic equation by the quadratic formula.

Because we used the method of completing the square to derive the quadratic formula, we no longer need it for solving quadratic equations. However, we will use completing the square later in the book to help graph circles and other kinds of equations.

Table P.5 on the next page summarizes our observations about which technique to use when solving a quadratic equation.

Table P.5 Determining the Most Efficient Technique to Use When Solving a Quadratic Equation

Description and Form of the Quadratic Equation	Most Efficient Solution Method	Example
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ can be factored easily.	Factor and use the zero-product principle.	$3x^2 + 5x - 2 = 0$ $(3x - 1)(x + 2) = 0$ $3x - 1 = 0$ or $x + 2 = 0$ $x = \frac{1}{3}$ $\quad x = -2$
$ax^2 + bx = 0$ The quadratic equation has no constant term. $(c = 0)$	Factor and use the zero-product principle.	$6x^2 + 9x = 0$ $3x(2x + 3) = 0$ $3x = 0$ or $2x + 3 = 0$ $x = 0$ $\quad 2x = -3$ $x = -\frac{3}{2}$
$ax^2 + c = 0$ The quadratic equation has no x-term. $(b = 0)$	Solve for x^2 and apply the square root property.	$7x^2 - 4 = 0$ $7x^2 = 4$ $x^2 = \frac{4}{7}$ $x = \pm\sqrt{\frac{4}{7}}$ $= \pm\frac{2}{\sqrt{7}} = \pm\frac{2}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \pm\frac{2\sqrt{7}}{7}$
$u^2 = d$; u is a first-degree polynomial.	Use the square root property.	$(x + 4)^2 = 5$ $x + 4 = \pm\sqrt{5}$ $x = -4 \pm \sqrt{5}$
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ cannot be factored or the factoring is too difficult.	Use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	$x^2 - 2x - 6 = 0$ $a = 1$ $\quad b = -2$ $\quad c = -6$ $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$ $= \frac{2 \pm \sqrt{4 + 24}}{2(1)}$ $= \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm \sqrt{4}\sqrt{7}}{2}$ $= \frac{2 \pm 2\sqrt{7}}{2} = \frac{2(1 \pm \sqrt{7})}{2}$ $= 1 \pm \sqrt{7}$

12 Solve radical equations.

Radical Equations

A **radical equation** is an equation in which the variable occurs in a square root, cube root, or any higher root. An example of a radical equation is

$$\sqrt{x} = 9.$$

We solve the equation by squaring both sides:

Squaring both sides eliminates the square root.

$$(\sqrt{x})^2 = 9^2$$
$$x = 81.$$

The proposed solution, 81, can be checked in the original equation, $\sqrt{x} = 9$. Because $\sqrt{81} = 9$, the solution is 81 and the solution set is $\{81\}$.

In general, we solve radical equations with square roots by squaring both sides of the equation. We solve radical equations with nth roots by raising both sides of the equation to the nth power. Unfortunately, if n is even, all the solutions of the equation raised to the even power may not be solutions of the original equation. Consider, for example, the equation

$$x = 4.$$

If we square both sides, we obtain

$$x^2 = 16.$$

Solving this equation using the square root property, we obtain

$$x = \pm\sqrt{16} = \pm 4.$$

The new equation $x^2 = 16$ has two solutions, -4 and 4. By contrast, only 4 is a solution of the original equation, $x = 4$. For this reason, **when raising both sides of an equation to an even power, always check proposed solutions in the original equation**.

Here is a general method for solving radical equations with nth roots:

Solving Radical Equations Containing *n*th Roots

1. If necessary, arrange terms so that one radical is isolated on one side of the equation.
2. Raise both sides of the equation to the nth power to eliminate the isolated nth root.
3. Solve the resulting equation. If this equation still contains radicals, repeat steps 1 and 2.
4. Check all proposed solutions in the original equation.

Extra solutions may be introduced when you raise both sides of a radical equation to an even power. Such solutions, which are not solutions of the given equation, are called **extraneous solutions** or **extraneous roots**.

EXAMPLE 12 Solving a Radical Equation

Solve: $\sqrt{2x - 1} + 2 = x$.

Solution

Step 1 Isolate a radical on one side. We isolate the radical, $\sqrt{2x - 1}$, by subtracting 2 from both sides.

$$\sqrt{2x-1} + 2 = x \quad \text{This is the given equation.}$$

$$\sqrt{2x-1} = x - 2 \quad \text{Subtract 2 from both sides.}$$

Step 2 Raise both sides to the *n*th power. Because n, the index, is 2, we square both sides.

$$\left(\sqrt{2x-1}\right)^2 = (x-2)^2$$

$$2x - 1 = x^2 - 4x + 4 \quad \text{Simplify. Use the formula } (A - B)^2 = A^2 - 2AB + B^2 \text{ on the right side.}$$

Step 3 Solve the resulting equation. Because of the x^2-term, the resulting equation is a quadratic equation. We can obtain 0 on the left side by subtracting $2x$ and adding 1 on both sides.

$$2x - 1 = x^2 - 4x + 4 \quad \text{The resulting equation is quadratic.}$$

$$0 = x^2 - 6x + 5 \quad \text{Write in general form, subtracting } 2x \text{ and adding 1 on both sides.}$$

$$0 = (x-1)(x-5) \quad \text{Factor.}$$

$$x - 1 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = 1 \qquad\qquad x = 5 \quad \text{Solve the resulting equations.}$$

Study Tip

Be sure to square *both sides* of an equation. Do *not* square each term.

Correct:

$$\left(\sqrt{2x-1}\right)^2 = (x-2)^2$$

Incorrect!

$$\cancel{\left(\sqrt{2x-1}\right)^2 = x^2 - 2^2}$$

Step 4 Check the proposed solutions in the original equation.

Check 1:	**Check 5:**
$\sqrt{2x-1}+2=x$	$\sqrt{2x-1}+2=x$
$\sqrt{2\cdot 1-1}+2\stackrel{?}{=}1$	$\sqrt{2\cdot 5-1}+2\stackrel{?}{=}5$
$\sqrt{1}+2\stackrel{?}{=}1$	$\sqrt{9}+2\stackrel{?}{=}5$
$1+2\stackrel{?}{=}1$	$3+2\stackrel{?}{=}5$
$3=1$, false	$5=5$, true

Thus, 1 is an extraneous solution. The only solution is 5, and the solution set is $\{5\}$.

Check Point 12 Solve: $\sqrt{x+3}+3=x$.

Exercise Set P.7

Practice Exercises

In Exercises 1–16, solve each linear equation.

1. $7x-5=72$

2. $6x-3=63$

3. $11x-(6x-5)=40$

4. $5x-(2x-10)=35$

5. $2x-7=6+x$

6. $3x+5=2x+13$

7. $7x+4=x+16$

8. $13x+14=12x-5$

9. $3(x-2)+7=2(x+5)$

10. $2(x-1)+3=x-3(x+1)$

11. $\frac{x+3}{6}=\frac{3}{8}+\frac{x-5}{4}$

12. $\frac{x+1}{4}=\frac{1}{6}+\frac{2-x}{3}$

13. $\frac{x}{4}=2+\frac{x-3}{3}$

14. $5+\frac{x-2}{3}=\frac{x+3}{8}$

15. $\frac{x+1}{3}=5-\frac{x+2}{7}$

16. $\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}$

Exercises 17–26 contain rational equations with variables in denominators. For each equation, **a.** *Write the value or values of the variable that make a denominator zero. These are the restrictions on the variable.* **b.** *Keeping the restrictions in mind, solve the equation.*

17. $\frac{1}{x-1}+5=\frac{11}{x-1}$

18. $\frac{3}{x+4}-7=\frac{-4}{x+4}$

19. $\frac{8x}{x+1}=4-\frac{8}{x+1}$

20. $\frac{2}{x-2}=\frac{x}{x-2}-2$

21. $\frac{3}{2x-2}+\frac{1}{2}=\frac{2}{x-1}$

22. $\frac{3}{x+3}=\frac{5}{2x+6}+\frac{1}{x-2}$

23. $\frac{2}{x+1}-\frac{1}{x-1}=\frac{2x}{x^2-1}$

24. $\frac{4}{x+5}+\frac{2}{x-5}=\frac{32}{x^2-25}$

25. $\frac{1}{x-4}-\frac{5}{x+2}=\frac{6}{x^2-2x-8}$

26. $\frac{1}{x-3}-\frac{2}{x+1}=\frac{8}{x^2-2x-3}$

In Exercises 27–42, solve each formula for the specified variable. Do you recognize the formula? If so, what does it describe?

27. $I=Prt$ for P

28. $C=2\pi r$ for r

29. $T=D+pm$ for p

30. $P=C+MC$ for M

31. $A=\frac{1}{2}h(a+b)$ for a

32. $A=\frac{1}{2}h(a+b)$ for b

33. $S=P+Prt$ for r

34. $S=P+Prt$ for t

35. $B=\frac{F}{S-V}$ for S

36. $S=\frac{C}{1-r}$ for r

37. $IR+Ir=E$ for I

38. $A=2lw+2lh+2wh$ for h

39. $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$ for f

40. $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$ for R_1

41. $f=\frac{f_1f_2}{f_1+f_2}$ for f_1

42. $f=\frac{f_1f_2}{f_1+f_2}$ for f_2

In Exercises 43–54, solve each absolute value equation or indicate the equation has no solution.

43. $|x-2|=7$

44. $|x+1|=5$

45. $|2x-1|=5$

46. $|2x-3|=11$

47. $2|3x-2|=14$

48. $3|2x-1|=21$

49. $2\left|4-\frac{5}{2}x\right|+6=18$

50. $4\left|1-\frac{3}{4}x\right|+7=10$

51. $|x+1|+5=3$

52. $|x+1|+6=2$

53. $|2x-1|+3=3$

54. $|3x-2|+4=4$

In Exercises 55–60, solve each quadratic equation by factoring.

55. $x^2-3x-10=0$

56. $x^2-13x+36=0$

57. $x^2=8x-15$

58. $x^2=-11x-10$

59. $5x^2=20x$

60. $3x^2=12x$

In Exercises 61–66, solve each quadratic equation by the square root property.

61. $3x^2 = 27$ **62.** $5x^2 = 45$

63. $5x^2 + 1 = 51$ **64.** $3x^2 - 1 = 47$

65. $3(x - 4)^2 = 15$ **66.** $3(x + 4)^2 = 21$

In Exercises 67–74, solve each quadratic equation by completing the square.

67. $x^2 + 6x = 7$ **68.** $x^2 + 6x = -8$

69. $x^2 - 2x = 2$ **70.** $x^2 + 4x = 12$

71. $x^2 - 6x - 11 = 0$ **72.** $x^2 - 2x - 5 = 0$

73. $x^2 + 4x + 1 = 0$ **74.** $x^2 + 6x - 5 = 0$

In Exercises 75–82, solve each quadratic equation using the quadratic formula.

75. $x^2 + 8x + 15 = 0$ **76.** $x^2 + 8x + 12 = 0$

77. $x^2 + 5x + 3 = 0$ **78.** $x^2 + 5x + 2 = 0$

79. $3x^2 - 3x - 4 = 0$ **80.** $5x^2 + x - 2 = 0$

81. $4x^2 = 2x + 7$ **82.** $3x^2 = 6x - 1$

Compute the discriminant of each equation in Exercises 83–90. What does the discriminant indicate about the number and type of solutions?

83. $x^2 - 4x - 5 = 0$ **84.** $4x^2 - 2x + 3 = 0$

85. $2x^2 - 11x + 3 = 0$ **86.** $2x^2 + 11x - 6 = 0$

87. $x^2 = 2x - 1$ **88.** $3x^2 = 2x - 1$

89. $x^2 - 3x - 7 = 0$ **90.** $3x^2 + 4x - 2 = 0$

In Exercises 91–114, solve each quadratic equation by the method of your choice.

91. $2x^2 - x = 1$ **92.** $3x^2 - 4x = 4$

93. $5x^2 + 2 = 11x$ **94.** $5x^2 = 6 - 13x$

95. $3x^2 = 60$ **96.** $2x^2 = 250$

97. $x^2 - 2x = 1$ **98.** $2x^2 + 3x = 1$

99. $(2x + 3)(x + 4) = 1$ **100.** $(2x - 5)(x + 1) = 2$

101. $(3x - 4)^2 = 16$ **102.** $(2x + 7)^2 = 25$

103. $3x^2 - 12x + 12 = 0$ **104.** $9 - 6x + x^2 = 0$

105. $4x^2 - 16 = 0$ **106.** $3x^2 - 27 = 0$

107. $x^2 = 4x - 2$ **108.** $x^2 = 6x - 7$

109. $2x^2 - 7x = 0$ **110.** $2x^2 + 5x = 3$

111. $\dfrac{1}{x} + \dfrac{1}{x + 2} = \dfrac{1}{3}$ **112.** $\dfrac{1}{x} + \dfrac{1}{x + 3} = \dfrac{1}{4}$

113. $\dfrac{2x}{x - 3} + \dfrac{6}{x + 3} = -\dfrac{28}{x^2 - 9}$

114. $\dfrac{3}{x - 3} + \dfrac{5}{x - 4} = \dfrac{x^2 - 20}{x^2 - 7x + 12}$

In Exercises 115–124, solve each radical equation. Check all proposed solutions.

115. $\sqrt{3x + 18} = x$ **116.** $\sqrt{20 - 8x} = x$

117. $\sqrt{x + 3} = x - 3$ **118.** $\sqrt{x + 10} = x - 2$

119. $\sqrt{2x + 13} = x + 7$ **120.** $\sqrt{6x + 1} = x - 1$

121. $x - \sqrt{2x + 5} = 5$ **122.** $x - \sqrt{x + 11} = 1$

123. $\sqrt{2x + 19} - 8 = x$ **124.** $\sqrt{2x + 15} - 6 = x$

Practice Plus

In Exercises 125–134, solve each equation.

125. $25 - [2 + 5x - 3(x + 2)] = -3(2x - 5) - [5(x - 1) - 3x + 3]$

126. $45 - [4 - 2x - 4(x + 7)] = -4(1 + 3x) - [4 - 3(x + 2) - 2(2x - 5)]$

127. $7 - 7x = (3x + 2)(x - 1)$

128. $10x - 1 = (2x + 1)^2$

129. $|x^2 + 2x - 36| = 12$

130. $|x^2 + 6x + 1| = 8$

131. $\dfrac{1}{x^2 - 3x + 2} = \dfrac{1}{x + 2} + \dfrac{5}{x^2 - 4}$

132. $\dfrac{x - 1}{x - 2} + \dfrac{x}{x - 3} = \dfrac{1}{x^2 - 5x + 6}$

133. $\sqrt{x + 8} - \sqrt{x - 4} = 2$

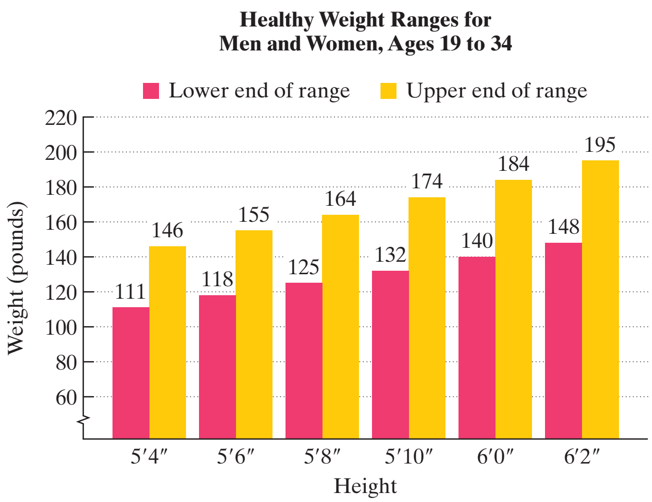
134. $\sqrt{x + 5} - \sqrt{x - 3} = 2$

In Exercises 135–136, list all numbers that must be excluded from the domain of each rational expression.

135. $\dfrac{3}{2x^2 + 4x - 9}$ **136.** $\dfrac{7}{2x^2 - 8x + 5}$

Application Exercises

The latest guidelines, which apply to both men and women, give healthy weight ranges, rather than specific weights, for your height. The further you are above the upper limit of your range, the greater are the risks of developing weight-related health problems. The bar graph shows these ranges for various heights for people between the ages of 19 and 34, inclusive.



Source: U.S. Department of Health and Human Services

The mathematical model

$$\frac{W}{2} - 3H = 53$$

describes a weight, W, in pounds, that lies within the healthy weight range shown by the bar graph on the previous page for a person whose height is H inches over 5 feet. Use this information to solve Exercises 137–138.

137. Use the formula to find a healthy weight for a person whose height is 5′6″. (*Hint:* $H = 6$ because this person's height is 6 inches over 5 feet.) How many pounds is this healthy weight below the upper end of the range shown by the bar graph on the previous page?

138. Use the formula to find a healthy weight for a person whose height is 6′0″. (*Hint:* $H = 12$ because this person's height is 12 inches over 5 feet.) How many pounds is this healthy weight below the upper end of the range shown by the bar graph on the previous page?

139. A company wants to increase the 10% peroxide content of its product by adding pure peroxide (100% peroxide). If x liters of pure peroxide are added to 500 liters of its 10% solution, the concentration, C, of the new mixture is given by

$$C = \frac{x + 0.1(500)}{x + 500}.$$

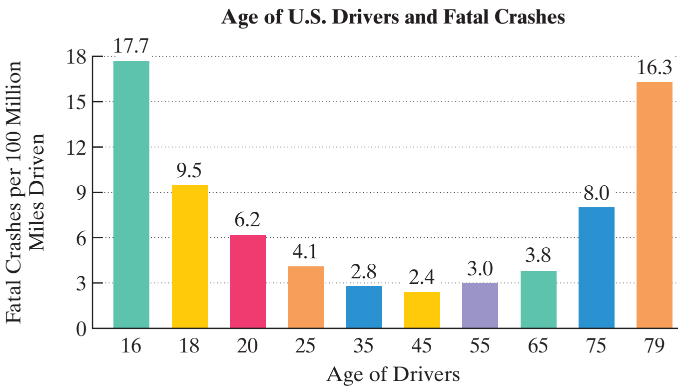
How many liters of pure peroxide should be added to produce a new product that is 28% peroxide?

140. Suppose that x liters of pure acid are added to 200 liters of a 35% acid solution.

a. Write a formula that gives the concentration, C, of the new mixture. (*Hint:* See Exercise 139.)

b. How many liters of pure acid should be added to produce a new mixture that is 74% acid?

A driver's age has something to do with his or her chance of getting into a fatal car crash. The bar graph shows the number of fatal vehicle crashes per 100 million miles driven for drivers of various age groups. For example, 25-year-old drivers are involved in 4.1 fatal crashes per 100 million miles driven. Thus, when a group of 25-year-old Americans have driven a total of 100 million miles, approximately 4 have been in accidents in which someone died.



Source: Insurance Institute for Highway Safety

The number of fatal vehicle crashes per 100 million miles, y, for drivers of age x can be modeled by the formula

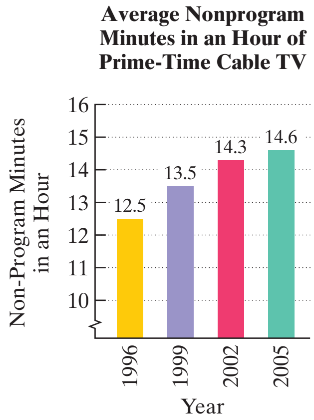
$$y = 0.013x^2 - 1.19x + 28.24.$$

Use the formula at the bottom of the previous column to solve Exercises 141–142.

141. What age groups are expected to be involved in 3 fatal crashes per 100 million miles driven? How well does the formula model the trend in the actual data shown by the bar graph in the previous column?

142. What age groups are expected to be involved in 10 fatal crashes per 100 million miles driven? How well does the formula model the trend in the actual data shown by the bar graph in the previous column?

By 2005, the amount of "clutter," including commercials and plugs for other shows, had increased to the point where an "hour-long" drama on cable TV was 45.4 minutes. The bar graph shows the average number of nonprogram minutes in an hour of prime-time cable television.



Source: Nielsen Monitor-Plus

The data can be modeled by the formula

$$M = 0.7\sqrt{x} + 12.5,$$

where M is the average number of nonprogram minutes in an hour of prime-time cable x years after 1996. Use the formula to solve Exercises 143–144.

143. Assuming the trend from 1996 through 2005 continues, use the model to project when there will be 15.1 cluttered minutes in every prime-time cable TV hour.

144. Assuming the trend from 1996 through 2005 continues, use the model to project when there will be 16 cluttered minutes in every prime-time cable TV hour.

Writing in Mathematics

145. What is a linear equation in one variable? Give an example of this type of equation.

146. Explain how to determine the restrictions on the variable for the equation

$$\frac{3}{x+5} + \frac{4}{x-2} = \frac{7}{x^2 + 3x - 6}.$$

147. What does it mean to solve a formula for a variable?

148. Explain how to solve an equation involving absolute value.

149. Why does the procedure that you explained in Exercise 148 not apply to the equation $|x - 2| = -3$? What is the solution set for this equation?

150. What is a quadratic equation?

151. Explain how to solve $x^2 + 6x + 8 = 0$ using factoring and the zero-product principle.

152. Explain how to solve $x^2 + 6x + 8 = 0$ by completing the square.

153. Explain how to solve $x^2 + 6x + 8 = 0$ using the quadratic formula.

154. How is the quadratic formula derived?

155. What is the discriminant and what information does it provide about a quadratic equation?

156. If you are given a quadratic equation, how do you determine which method to use to solve it?

157. In solving $\sqrt{2x - 1} + 2 = x$, why is it a good idea to isolate the radical term? What if we don't do this and simply square each side? Describe what happens.

158. What is an extraneous solution to a radical equation?

Critical Thinking Exercises

Make Sense? *In Exercises 159–162, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

159. The model $P = -0.18n + 2.1$ describes the number of pay phones, P, in millions, n years after 2000, so I have to solve a linear equation to determine the number of pay phones in 2006.

160. Although I can solve $3x + \frac{1}{5} = \frac{1}{4}$ by first subtracting $\frac{1}{5}$ from both sides, I find it easier to begin by multiplying both sides by 20, the least common denominator.

161. Because I want to solve $25x^2 - 169 = 0$ fairly quickly, I'll use the quadratic formula.

162. When checking a radical equation's proposed solution, I can substitute into the original equation or any equation that is part of the solution process.

In Exercises 163–166, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

163. The equation $(2x - 3)^2 = 25$ is equivalent to $2x - 3 = 5$.

164. Every quadratic equation has two distinct numbers in its solution set.

165. The equations $3y - 1 = 11$ and $3y - 7 = 5$ are equivalent.

166. The equation $ax^2 + c = 0$, $a \neq 0$, cannot be solved by the quadratic formula.

167. Find b such that $\frac{7x + 4}{b} + 13 = x$ will have a solution set given by $\{-6\}$.

168. Write a quadratic equation in general form whose solution set is $\{-3, 5\}$.

169. Solve for C: $V = C - \frac{C - S}{L}N$.

170. Solve for t: $s = -16t^2 + v_0 t$.

Preview Exercises

Exercises 171–173 will help you prepare for the material covered in the next section.

171. Jane's salary exceeds Jim's by \$150 per week. If x represents Jim's weekly salary, write an algebraic expression that models Jane's weekly salary.

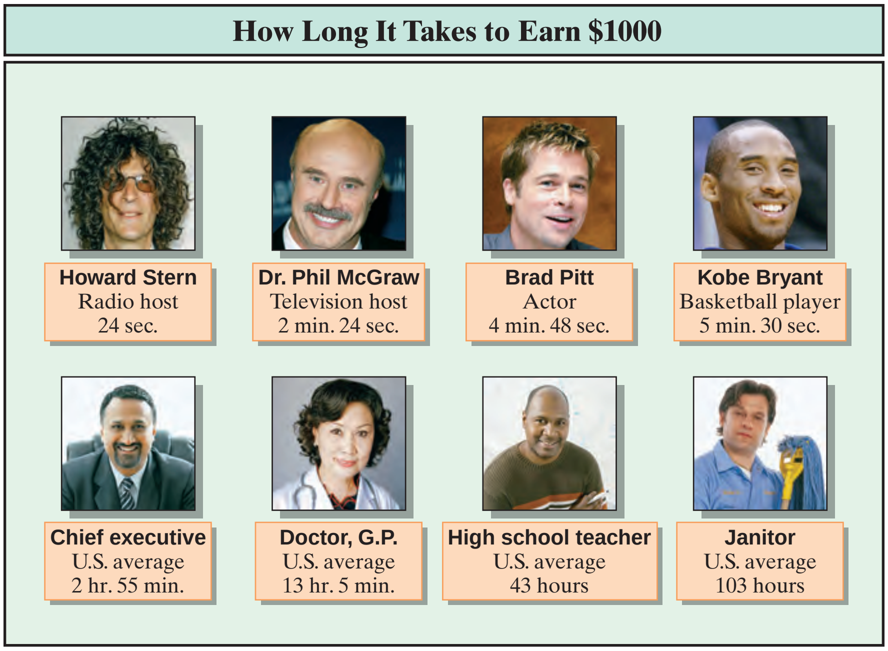
172. A long-distance telephone plan has a monthly fee of \$20 with a charge of \$0.05 per minute for all long-distance calls. Write an algebraic expression that models the plan's monthly cost for x minutes of long-distance calls.

173. If the width of a rectangle is represented by x and the length is represented by $x + 200$, write a simplified algebraic expression that models the rectangle's perimeter.

Section P.8 Modeling with Equations

Objective

1. Use equations to solve problems.



Source: Time

In this section, you'll see examples and exercises focused on how much money Americans earn. These situations illustrate a step-by-step strategy for solving problems. As you become familiar with this strategy, you will learn to solve a wide variety of problems.